Communications

R.J. Marks II Class Notes
Professor Hal Sabbaugh
Text: Wozencraft & Jacobs
Rose-Hulman Institute of Technology (1971)

COMMUNICATION SYSTEMS

COMMUNICATION SYSTEMS

3-19-71

MON 2.12, 2.23, 2.24 — COMM PROB " EX ON 78

(3.4 3.5) 3.4 3.6 3.7 3.8 CORR. REC.

ALL OF CHAPT. 4 MATCHED FILTR

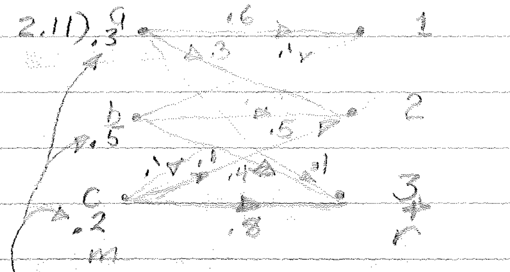
SKIP CHAPT. 5, 6

CHAPT 7: 1, 2

CHAPT 8; 1, 2, 3. 5

LECTURE

2.11)



APRIORI PROB = PROB. THAT a, b, c WERE SENT AFTER HEARING A 1, 2, 3

$P[m_k]$ = APRIORI PROB. OF k^{TH} MESSAGE

$P[r_j/m_k]$ = CONDITIONAL PROB OF RECEIVING THE j^{TH} r GIVEN OR CONDITIONED UPON TRANSMITTING THE k^{TH} MESSAGE

$(P[m_k]P[r_j|m_k] = P[m_k,r_j] = P[m_k/r_j]P[r_j$

$P[m_k, r_j]$ = PROB. OF THE "JOINT EVENT m_k TRANSMITTED AND r_j IS RECEIVED (APOSTERORI (CONDITIONAL) PROB)

BAYES RULE

$$P[m_k|r_j] = \frac{P[m_k r_j]}{P[r_j]}$$

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COMM PROB
" EX ON 78

MON 2.12, 2.23, 2.24

(3.4 3.3) 3.4 3.6 3.7 3.8
CORR. REC.

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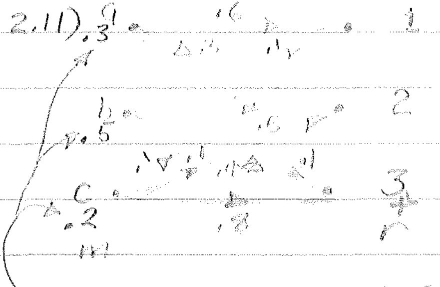


APRIORI PROB = PROB. THAT a, b, c WERE SENT AFTER HEARING A 1, 2, 3

$P[m_k]$ = APRIORI PROB. OF k^{th} MESSAGE

$P[r_j/m_k]$ = CONDITIONAL PROB OF RECEIVING THE j^{TH} r GIVEN OR CONDITIONED UPON TRANSMITTING THE K^{TH} MESSAGE

$$\left(P[m_k]P[r_j|m_k] = P[m_k, r_j] \overset{"}{=} P[m_k/r_j]P[r_j]\right.$$

$P[m_k, r_j]$ = PROB. OF THE "JOINT EVENT m_k TRANSMITTED AND r_j IS RECEIVED (APOSTERORI PROB)
(CONDITIONAL)

BAYES RULE

$$P[m_k|r_j] = \frac{P[m_k r_j]}{P[r_j]}$$

$$P[m_K|\underset{\substack{\uparrow\\ \text{FIXED}}}{r_j}] = \frac{P[m_K,r_j]}{\underset{\substack{\uparrow\\ \text{FIXED}}}{P[r_j]}}$$

WANT VALUE OF m_K TO MAX. $P[m_K|r_j]$

$$\frac{P[m_K,r_j]}{P[r_j]} = \frac{P[m_K]\,P[r_j/m_K]}{P[r_j]}$$

MAXIMIZING $\Rightarrow$ MAXIMIZING

$\underline{P[m_K]\,P[r_j/m_K]}$

$P[r_j] = P[r_j,m_1] + P[r_j,m_2] + P[r_j,m_3]$

$P[1] = P[1,a] + P[1,b] + P[1,c]$

$= (.3)(.6) + (.5)(.1) + (.2)(.1)$

$P[2] = P[2,a] + P[2,b] + P[2,c]$

$= (.3)(.3) + (.5)(.5) + (.2)(.1)$

$= .36$

$P[3] = .39$

$P[1,a] = .18$; $P[1,b] = .05$ $P[1,c] = .02$

$P[2,a] = .09$ $P[2,b] = .25$ $P[2.c] = .02$

$P[3,a] = .03$ $P[3,b] = .20$ $P[3,c] = .16$

RECEIVER DECISION RULE

$\hat{m}(1) = a$ (.18)

$\hat{m}(2) = b$ (.25)

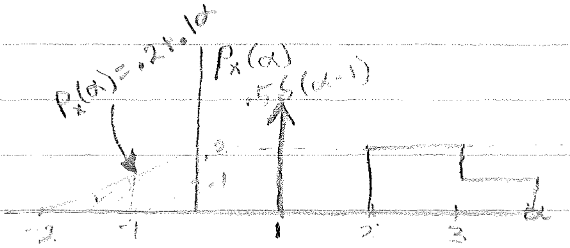
$\hat{m}(3) = b$ (.20)

b) $P[C] = .18 + .25 + .20 = .63 \Rightarrow P[E] = .37$

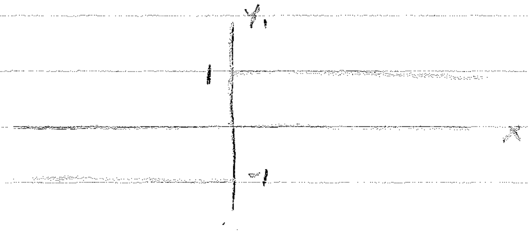
c) b DOMINATES $\Rightarrow P[E] = .50$

$\hat{m}(1,2,3) = b$

2.18)

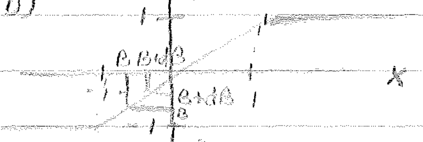


a)

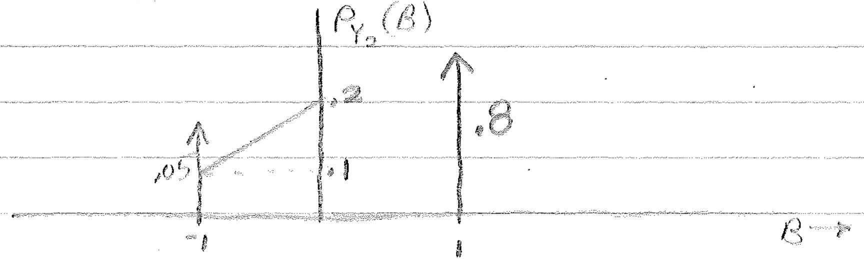


$$P_{Y_1}(\beta)=.2\delta(\beta+1)+.8\delta(\beta-1)$$

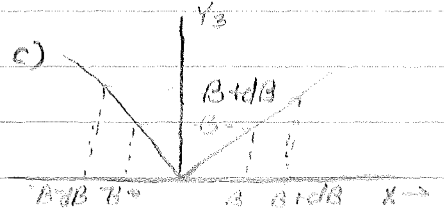
b) FOR $-1<\beta<0$, $P_{Y_2}(\beta)=.2+.1\beta$

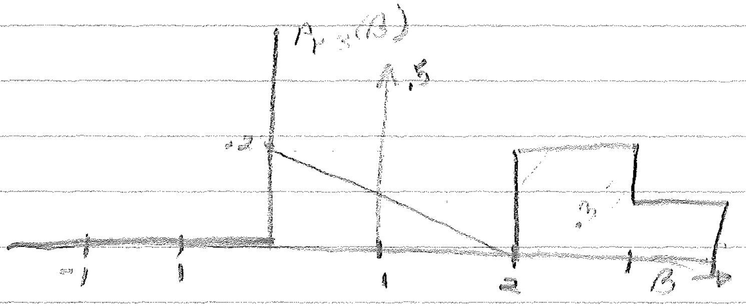


$$P_{Y_2}(\beta)=.05\delta(\beta+1)+.8\delta(\beta-1)+\overbrace{.2+.1\beta}^{-1<\beta<0}$$

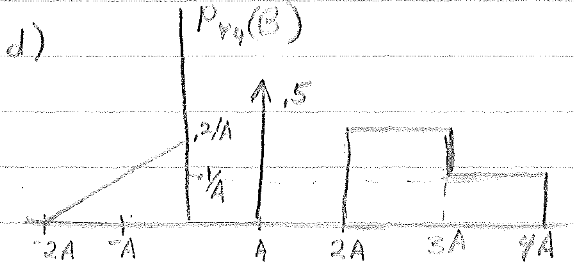


c)





d)



e) $P_{Y_5}(B) = .5\delta(B-1) + .1\delta(B-2) + .15\delta(B-3)$
$+ .05\delta(B-4) + .0875(B-4)$
$+ .1\delta(B+1) + .0125(B+2)$

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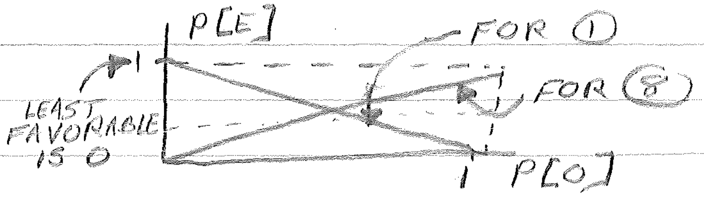
2.12) $(a,b,c) = \overset{①}{(0,0,0)}, \overset{②}{(001)}, \ldots, \overset{⑧}{(111)}$

8 PROB. OF MAKING CORR. CHOICE

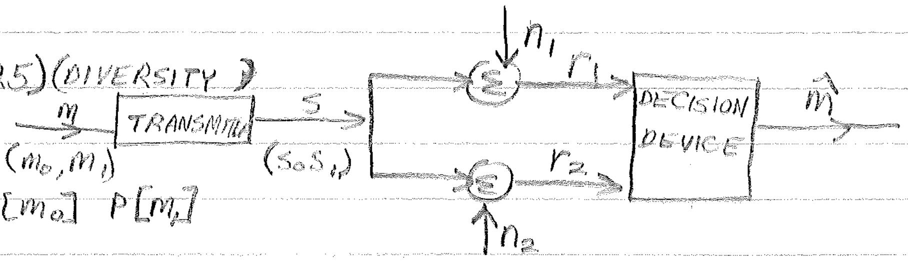
HINT: $(0,0,0)$ $P[C] = P[$0 WAS TRANSM.$]$

$P[E_1] = 1 - P[0]$



$P[E_8] = P[0]$

2.25) (DIVERSITY)



$s_0 = (-1)^0 \sqrt{E} = \sqrt{E}$ $\quad s_1 = (-1)^1 \sqrt{E} = -\sqrt{E}$

$\exists E = \text{ENERGY} \propto V^2$

$r_1 = s + n_1 \; ; \; r_2 = s + n_2$

BAYES MIXED RULE:

$$p_r(\rho/m)\, P[m] = P[m/r=\rho]\, p_r(\rho)$$

$$= p_r(\rho, m)$$ (JOINT DISTRIBUTION FUNCTION OF RANDOM EVENT r & EVENT m)

WANT TO MAXIMIZE A'POSTERORI PROB:

$$P[m_i/r=\rho] = \frac{p_r(\rho/m_i)\, P[m_i]}{p_r(\rho)}$$

$$P[m_0/r=\rho] > P[m_i/r=\rho]$$

$$\Rightarrow p_r(\rho/m_0)\, P[m_0] > p_r(\rho/m_i)\, P[m_i]$$

$r_i = s_i + n \Rightarrow p_r(\rho/m_i) = ?$

$$P[\rho < r < \rho + d\rho \,/\, m = m_i]$$

$$= P[\rho < s_i + n < \rho + d\rho \,/\, m_i]$$

$$= P[\rho - s_i < n < \rho - s_i + d\rho \,/\, m = m_i]$$

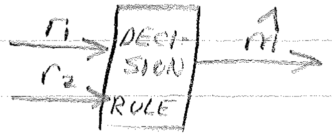
$$= p_n(\rho - s_i / m = m_i)\, d\rho$$

m_i IND OF $n \Rightarrow$

$$= p_n(\rho - s_i)\, d\rho$$

$$\boxed{\therefore p_r(\rho / m = m_i) = p_n(\rho - s_i)}$$

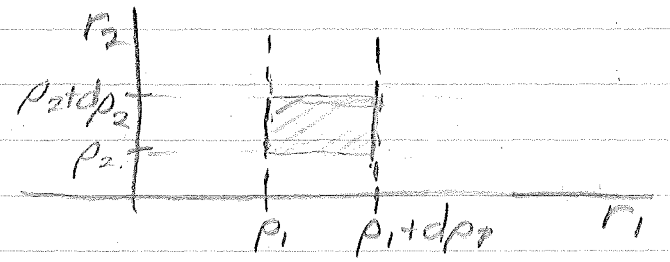
WE MUST EXAMINE THE TWO VARIABLE CASE



WANT $P[m_0 | r_1 = \rho_1, r_2 = \rho_2]$

$$P[m_0 | r_1 = \rho_1, r_2 = \rho_2] > P[m_1 | r_1 = \rho_1, r_2 = \rho_2]$$

$$\Rightarrow p_{r_1,r_2}(\rho_1 \rho_2 | m_0) P[m_0] > p_{r_1 r_2}(\rho_1 \rho_2 | m_1) P[m_1]$$



BY ANALOGY

$$p_{r_1 r_2}(\rho_1 \rho_2 | m_i) = p_{n_1 n_2}(\rho_1 - s_i, \rho_2 - s_2 | m_i)$$

(n_1 & n_2 INP FROM m_i

$$= p_{n_1}(\rho_1 - s_i) p_{n_2}(\rho_2 - s_i)$$

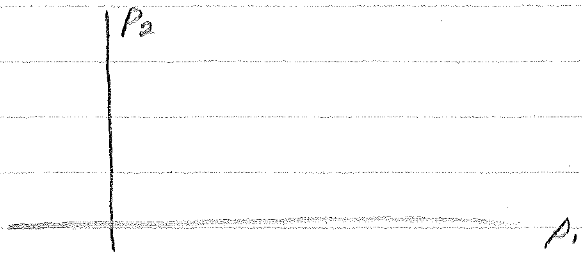
mean $= s^i$

$$p_{n_i}(\rho_1 - s_i) = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-(\rho_1 - s_i)^2 / 2\sigma_1^2}$$

$$p_{n_2}(\rho_2 - s_i) = \frac{1}{\sqrt{2\pi}\sigma_2} e^{-(\rho_2 - s_i)^2 / 2\sigma_2^2}$$

$$p_{r_1 r_2}(\rho_1 \rho_2 | m_0) P[m_0] = \frac{P[m_0]}{2\pi\sigma_1\sigma_2} e^{-\left(\frac{(\rho_1 - s_0)^2}{2\sigma_1^2} + \frac{(\rho_2 + s_0)^2}{2\sigma_2^2}\right)}$$

$$p_{r_1 r_2}(\rho_1 \rho_2 / m_1] P[m_1] = \frac{P[m_1]}{2\pi\sigma_1\sigma_2} e^{-\left[\frac{(\rho_1 - s_1)^2}{2\sigma_1^2} + \frac{(\rho_2 - s_0)^2}{2\sigma_2^2}\right]}$$



Let $\rho_1 = \rho_2$

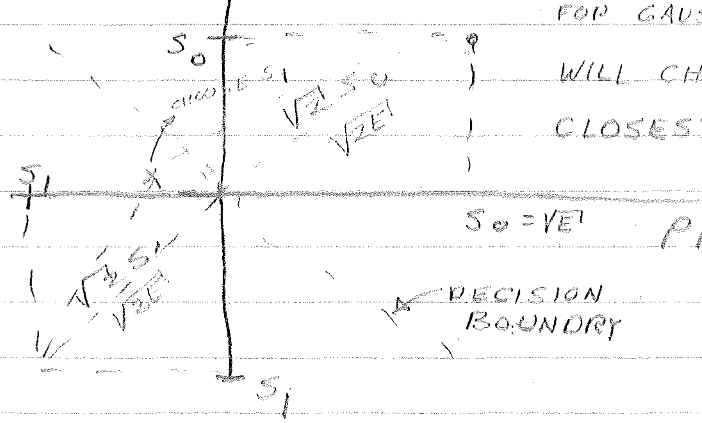
$$P_{r_1 r_2}(\rho_1, \rho_2 / m_0) P(m_0] = \frac{P(m_0)}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}\left[(\rho_1 - s_0)^2 + (\rho_2 - s_0)^2\right]}$$

$$P_{r_1 r_2}(\rho_1 \rho_2 / m_1) P[m_1] = \frac{P[m_1]}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}\left[(\rho_1 - s_1)^2 + (\rho_2 - s_1)^2\right]}$$

ALL THAT IS NEEDED IS EXPONENENS

$$(\rho_1 - s_0)^2 + (\rho_2 - s_0)^2 > (\rho_1 - s_1)^2 + (\rho_2 - s_1)^2$$

$P[m_0] = P[m_1]$; $s_0 = -s_1$



FOR GAUSSIAN NOISE,
WILL CHOOSE SIGNAL
CLOSEST TO VOLTAGE

THE FARTHER APART THE POINTS,
THE BETTER THE DECISION

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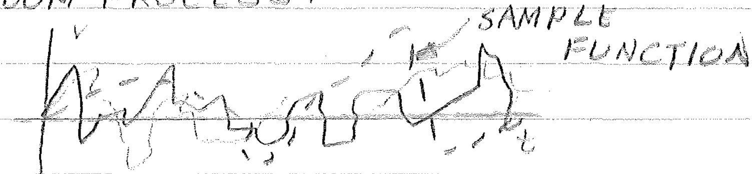
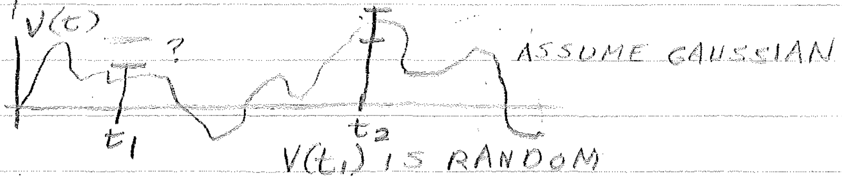
CHAPT. 3

RANDOM PROCESS:





ASSUME GAUSSIAN

$V(t_1)$ IS RANDOM

$\rho_{x(t)}(\underset{\uparrow}{\alpha}, t_1)$

SINGLE VAR. = V at t_1

$\rho_{x(t_1),x(t_2)}(\alpha_{t_1}, \beta_{t_2})$

$\vdots$

$\rho_{x(t_1)\ldots x(t_n)}(\overbrace{\alpha_1, \ldots, \alpha_n}^{\rho})$

IF EACH IS GAUSSIAN ρ IS GAUSSIAN

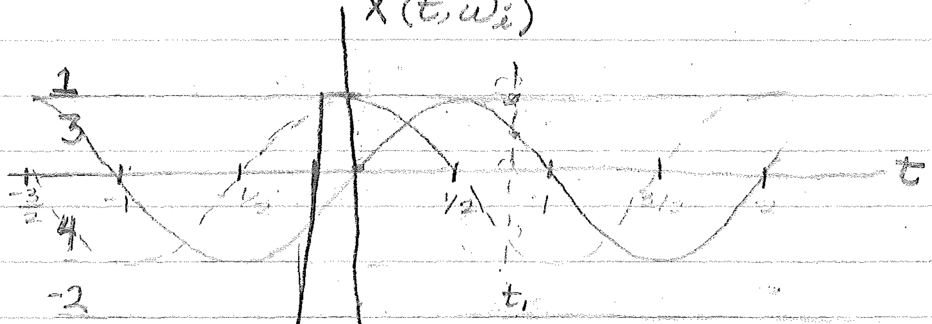
IF P IS DEPENDENT ONLY ON Δt,
SYSTEM IS STATIONARY

NEED ONLY $\overline{X^2}$ & $\overline{X}$ TO DEFINE
GAUSSIAN PRODUCT

READ: 3.1, $\tilde{3.2}$, 3.4, 3.5

(3.1) $x(\omega_1, t) = 1 \quad x(\omega_3\, t) = \sin \pi t$

$x(\omega_2\, t) = -2 \quad x(\omega_4, t) = \cos \pi t$



a) $P[\omega_i] = \frac{1}{4} \ \forall i$

$$P_{x(t_1)}(\alpha) = \frac{1}{4}\delta(\alpha-1) + \frac{1}{4}(\delta+2) + \frac{1}{4}\delta(\alpha - \sin\pi t_1) + \frac{1}{4}\delta(\alpha - \cos\pi t_1)$$

$P_{x(t_1)}(\alpha)$ CHANGES WITH TIME $\Rightarrow$ NOT STATIONARY

b) $\bar{x}(t) = \int_{-\infty}^{\infty} \alpha\, P_{x(t)}(\alpha)\, d\alpha$

$$= \frac{1}{4}(1 - 2 + \sin\pi t + \cos\pi t)$$

$$\overline{x(t_1)\,x(t_2)} = \int_{-\infty}^{\infty}\int \alpha\,\beta\, P_{x(t_1)\,x(t_2)}\, d\alpha\, d\beta$$

$= R(t_1, t_2) =$ CORRELATION FUNC.

TO CALCULATE: $P_{x(t_1)\,x(t_2)}(\alpha, \beta)$

$$= P_{\bar{x}(t)}(\bar{\alpha}) = p_{x(t_1)}(\alpha)\, p_{x(t_2)}[\beta / x(t_1 = \alpha]$$

WHAT IS $P_{x(t_2)}(\beta / x(t_1) = \alpha)$?

LET $t_1 = \frac{1}{4}$, THEN ONLY VALUES OF α ARE $-1, -2, \pm .707$

THEN, IF $\alpha = 1$, $P_{x(t_2)}(\beta / x(t_1) = 1)$

$= \delta(\beta - 1)$ (MUST STAY ON SAME FUNCTION)

THEN, IF $\alpha = -2$, $P_{x(t_2)}(\beta / x(t_1 = -2)$

$= \delta(\beta + 2)$

THEN, IF $\alpha = .707$, $P_{x(t_2)}(\beta / x(t_1) = .707)$

$= \delta(\beta - \sin\pi t_1)$

THEN, IF $\alpha = -.707$, $P_{x(t_2)}(\beta / x(t_1) = -.707)$

$= \delta(\beta - \cos \pi t_2)$

$\Rightarrow \overline{x(t_1)(x(t_2)} = \overline{x(-\tfrac{1}{4})\,x(-\tfrac{1}{4})}$

$= \int\int_{-\infty}^{\infty} \alpha \cdot \beta \cdot \tfrac{1}{4}[\delta(\alpha-1) + \delta(\alpha+2)$

$+ \delta(\alpha+.707) + \delta(\alpha-.707]$

$P[\beta / x(-\tfrac{1}{4}) = \alpha)\, d\alpha\, d\beta$

$= \int_{-\infty}^{\infty}[\beta \cdot \tfrac{1}{4}[p_{x(t_2)}(\beta / x(-\tfrac{1}{4}) = 1)$

$- 2p_{x(t_2)}(\beta / x(-\tfrac{1}{4}) = -2)$

$-.707\, p_{x(t_2)}(\beta / x(-\tfrac{1}{4}) = -.707)$

$+.707\, p_{x(t_2)}(\beta / x(-\tfrac{1}{4}) = .707)]\, d\beta$

$= \int_{-\infty}^{\infty} \beta\, \tfrac{1}{4}[\delta(\beta-1) - 2\delta(\beta+2)$

$-.707\delta(\beta - \sin \pi t_2 + .707\delta(\beta - \cos \pi t_2)$

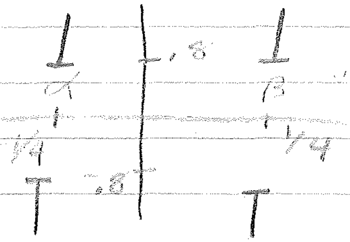
$\therefore \overline{x(t_1)x(t_2)} = \tfrac{1}{4} + 1 - \tfrac{.707}{4} \sin \pi t_2$

$+ \tfrac{.707}{4} \cos \pi t_2 = \tfrac{5}{4} + \tfrac{.707}{4}(\cos \pi t_2 - \sin \pi t_2)$

FOR $t_1 = -1/4$

CAN'T PUT AS $f(t_2 - t_1) \Rightarrow$ NOT SECOND ORDER STATIONARY

c)



$P_{ROB} = \int_{-.8}^{.8}\int_{-.8}^{.8} P_{x(-\frac{1}{4})}(\alpha) P_{x(\frac{1}{4})}(\beta / x(-\tfrac{1}{4}) = \alpha)\, d\alpha\, d\beta$

$= \int\int_{-.8}^{.8} \tfrac{1}{4}[\delta(\alpha-1)\delta(\beta-1)$

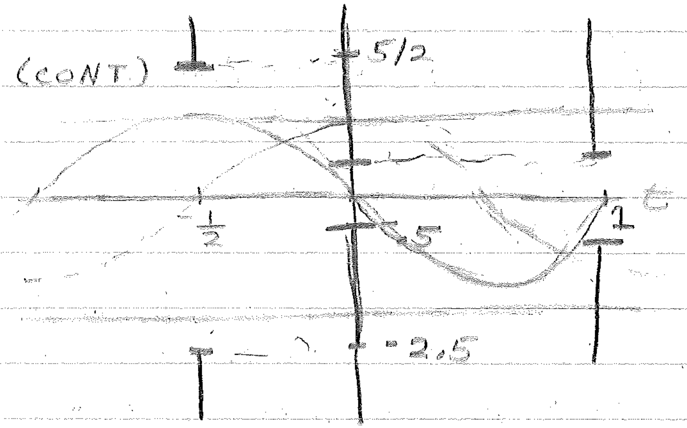
$+ \delta(\alpha+2)\,\delta(\beta+2)$

$+ \delta(\alpha+.707)\,\delta(\beta-.707) + \delta(\alpha-.707)\,\delta(\beta-.707)]\, d\alpha$

$\therefore P = \tfrac{1}{2}$

(CONT.)

c) (CONT)



$\rho = 2/3 = \frac{8}{12}$

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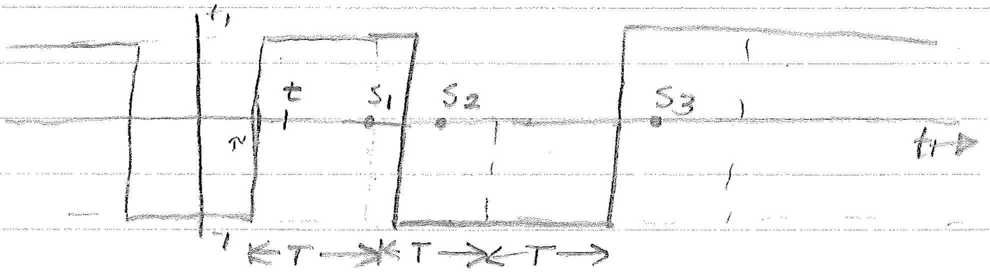
3-12)

$$x(t) = \sum_{i=-\infty}^{\infty} w_i\, u(t - iT - \overset{R.V.}{\tau})$$



w_i, THE VALUE OF $x(t)$ IN THE i^{TH} INTERVAL, IS IND. OF w_j, THE VALUE OF $x(t)$ IN j^{TH} INTERVAL

$\therefore$ WHEN $|t-S_3| > T$, THE CORRELATION FUNCTION

$$R_x(t,S_3) = \overline{x(t)x(S_3)} = \frac{1}{4}\underbrace{(1-1-1+1)} = 0$$

PRODUCT OF ALL POSSIBLE PAIRS OF VALUES OF $X(t)$ AND $X(S_3)$

WHEN t AND S ARE IN SAME INTERVAL

EX) LET $S = S_1$

$$\Rightarrow \underline{x(t) = x(S_1)} = \pm 1$$

$$\therefore \overline{x(t)x(S_1)} = R_x(t,S) = 1$$

$$\therefore \overline{x(t)x(S)} = 1 = P[t \,\&\, S \text{ ARE IN SAME INTERVAL}$$
$$+ 0 \cdot P[t \,\&\, S \text{ ARE NOT IN SAME INTERVAL}$$
$$= P[t \,\&\, S \text{ ARE IN SAME INTERVAL}]$$

FROM THE FIGURE, t & S WILL LIE IN THE SAME T SEC. INTERVAL IFF

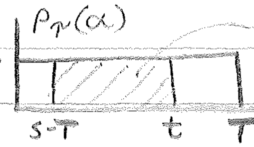
$S - T < \tau < t$. HOLDS FOR

$S > t$ (FOR EX., IN THE FIGURE)

$$P[S - T < \tau < t] = (t - S + T)\frac{1}{T} = 1 - \frac{S-t}{T}$$

$$= 1 - \frac{U}{T} \quad (U = S - t)$$

FOR $S > t$

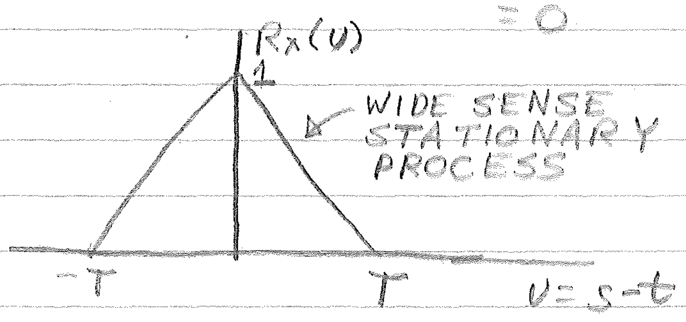


LET $S < T$, THEN S & t BELONG TO THE SAME INTERVAL IFF

$t - T < \tau < S$

$$\Rightarrow P[t - T < \tau < S] = 1 + \frac{U}{T} \quad (U = S - T)$$

$$\therefore R_x(t,S) = R_x(S-t) = 1 + \frac{U}{T} \qquad U < 0$$
$$= 1 - \frac{U}{T} \qquad U > 0$$
$$= 0 \qquad |U| > T$$



13

AMBIGUITY FUNCTIONS

$x(t)$

$x_\tau(t) = x(t+\tau)$

DEFINE A SIGNAL SPACE

$\{x(t), y(t), z(t), \underset{\S}{\zeta}(t) \ldots\}$

PLUS A 'METRIC' OR 'DISTANCE' BETWEEN SIGNALS

$$\overset{\text{DISTANCE}}{d}(x, y) = \int_{-\infty}^{\infty} |x(t) - y(t)|^2 dt$$

$$d(x, x_\tau) = \int_{-\infty}^{\infty} |x(t) - x_\tau(t|^2 dt$$

$$= \int_{-\infty}^{\infty} [x(t) - x(t+\tau)]$$

$$[x(t) - x(t+\tau] dt$$

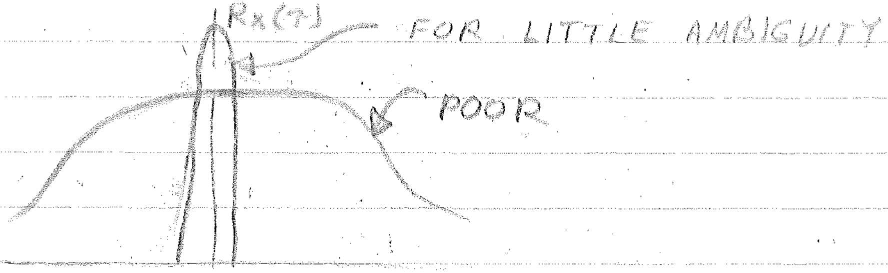
$$= \int_{-\infty}^{\infty} [x(t)\overset{\text{POWER}}{x}(t) + x(t+\tau)x(t+\tau)] dt$$

$$- \int_{-\infty}^{\infty} [x(t)x(t+\tau) + x(t+\tau)x(t)] dt$$

$$\left(R_x(u) = \int_{-\infty}^{\infty} x(t)x(t+u) dt \right)$$

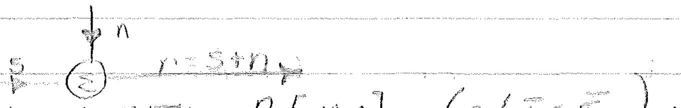
$$\Rightarrow d(x, x_\tau) = R_x(0) + R_x(0) - R_x(\tau) - R_x(\tau)$$

$$= 2(R_x(0) - R_x(\tau))$$



5-10-71

4-1)



MUST MAXIMIZE: $P[m_i]p_r(\rho/\bar{s}=\bar{s}\,)\;;\;i=01$

FOR $i=k$, $\Longrightarrow \hat{m}=m_k$

NOW, $P[m_i]=1/2$ FOR $i=0,1$

i.e. $P[m_i]$ IS INDEPENDENT OF i.

HENCE, CHOOSE $\hat{M}=m_k$ IF

$p_r(\bar{\rho}/s=s_i)$ IS MAX FOR $i=k$.

$r_{2i}=n+s_{2i}$

$\bar{r}=n\begin{bmatrix}1\\1\end{bmatrix}+S$ OR $r_{1i}=n+s_{1i}$

FORGET VECTOR NATURE: $r=n+s$

GIVEN p_n, WHAT IS $p_r(\rho/s=s_i)$

$n=r-s$

$p_r(\rho/s=s_i)=p_n(\rho-s_i/s=s_i)$

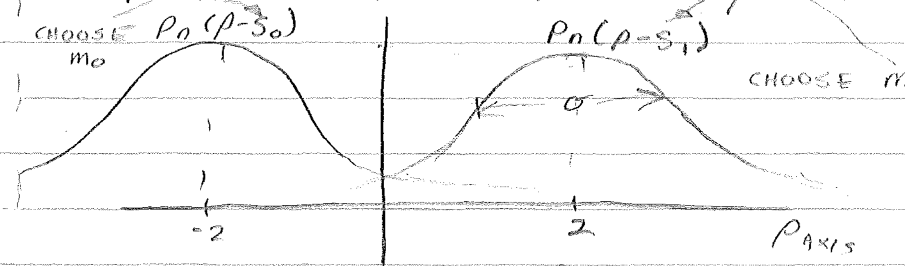
$=p_n(\rho-s_i)$ (n IS STATISTICALLY IND. OF s)

BECAUSE NOISE IS 0-MEAN GAUSSIAN

$p_n(\rho-s_i)=\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(\frac{-1}{2\sigma^2}(\rho-s_i)^2\right)$

$\Rightarrow p_n(\rho-s_0)=\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(\frac{-1}{2\sigma^2}(\rho+2)^2\right)$

$p_n(\rho-s_1)=\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(\frac{-1}{2\sigma^2}(\rho-2)^2\right)$



$p[C/m_0]=\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{0}\exp\left(\frac{-1}{2\sigma^2}(\rho+2)^2\right)d\rho$

$=\frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{2}\exp\left(\frac{-\sigma^2}{2\sigma^2}\right)d\sigma$

(CONT)

$$P[C/m_0] = \frac{1}{2} + \frac{1}{\sqrt{2\pi}\,\sigma} \int_0^2 \exp \frac{-\delta^2}{2\sigma^2} d\delta$$

$$P[C/m_1] = \frac{1}{2} + \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-2}^0 \exp \frac{-\delta^2}{2\sigma^2} d\delta$$

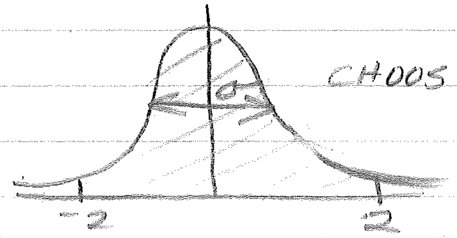
$$\Rightarrow P[c] = P[C/m_{0}] + P[C/m_1]\, P[m_i]$$

$$= \frac{1}{2} + \frac{1}{2}\frac{1}{\sqrt{2\pi}\,\sigma} \int_{-2}^{2} \exp\left(\frac{-\rho}{2\sigma^2}\right) d\rho$$

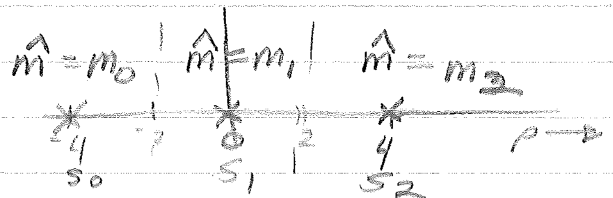
$$\overline{P[\varepsilon]} = .01 \Rightarrow P[C] = .99$$

$$\Rightarrow .99 = \frac{1}{2} + \frac{1}{2}\frac{1}{\sqrt{2\pi}\,\sigma} \int_{-2}^{2} \exp \frac{-\delta^2}{2\sigma^2} d\delta$$

$$\text{or } .98 \frac{1}{\sqrt{2\pi}\sqrt{\sigma^2}} \int_{-2}^{2} \exp\left(\frac{-\delta^2}{2\sigma^2} d\delta\right.$$



CHOOSE $\sigma \Rightarrow \int_{-2}^{2} \rho = 98\%$ OF WHOLE THING



FOR OPTIMUM RECIEVER WITH

$$P[m_i] = \frac{1}{3}\ \forall\ i$$

$$P[\varepsilon] = 1 - \frac{1}{3}\left[P(C/m_0) + P[C/m_1] + P[C/m_2]\right]$$

INVESTIGATE

$$p_n(\rho - s_0) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(\frac{-1}{2\sigma^2}(\rho+4)^2\right) = p_r(\rho/s = s_0)$$

$$p_n(\rho - s_1) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(\frac{-1}{2\sigma^2}\rho^2\right) = p_r(\rho/s = s_1)$$

$$p_n(\rho - s_2) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(\frac{-1}{2\sigma^2}(\rho-4)^2\right) = p_r(\rho/s = s_2)$$

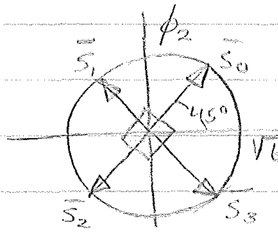
b) $p_n(\rho - s_i) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp \frac{-(\rho - 1 - s_i)^2}{2\sigma^2}$

⇒ SHIFTED TO RIGHT ONE UNIT

(NO CHANGE IN $P[\varepsilon]$)

READ APPENDIX 4 A

4-2)



(ϕ_1 ⅋ ϕ_2 ARE ORTHOGONAL $\int_{-\infty}^{\infty} \phi_1 \phi_2 \; 0$
$\int_{-\infty}^{\infty} \phi_1^2(t)dt = \int_{-\infty}^{\infty} \phi_2(t)dt =$ ENERGY
IF $E = 1$, ϕ_1 ⅋ ϕ_2 ARE ORTHONORMAL)

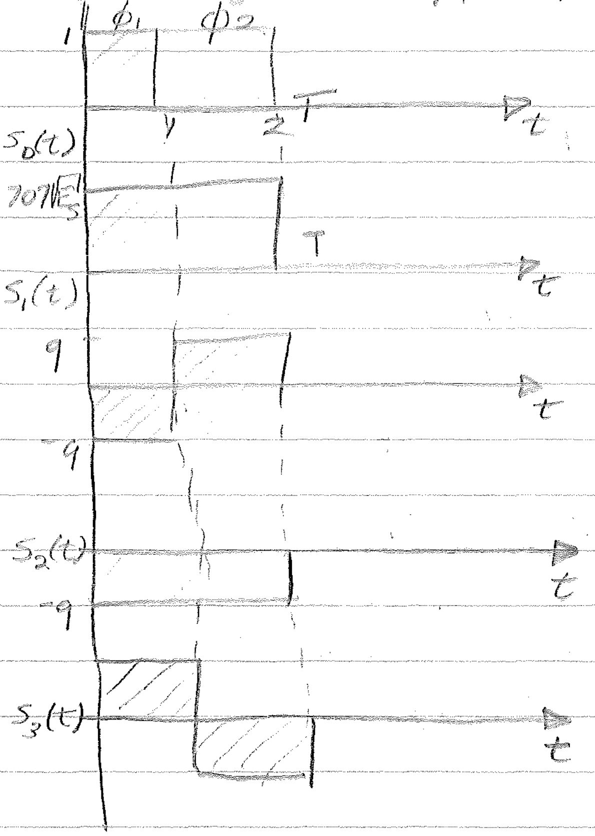
ENERGY $= E_S$

$\bar{S}_0 = \sqrt{E_S}\,(.707\phi_1 + .707\phi_2)$

$S_1 = \sqrt{E_S}\,(-.707\bar{\phi}_1 + .707\bar{\phi}_2)$

$S_2 = \sqrt{E_S}\,(-.707\bar{\phi}_1 - .707\bar{\phi}_2)$

$S_3 = \sqrt{E_S}\,(.707\phi_1 - .707\bar{\phi}_2)$



$q = .707\sqrt{E_S}$

$\int S_i^2(t) = E$

S_0 ORTH. TO S_1

S_0 " " S_3

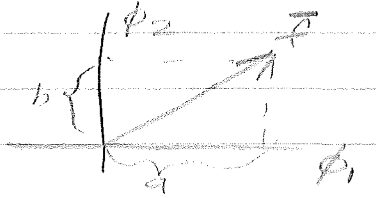
$S_1 = -S_3$

$S_0 = -S_2$

$f(t) \Longrightarrow \bar{f}$

$\bar{\phi}_1 = \phi_1(t)$; $\phi_2(t) = \bar{\phi}_2$ ORTHONORMAL

$\bar{f} = a\bar{\phi}_1 + b\bar{\phi}_2$

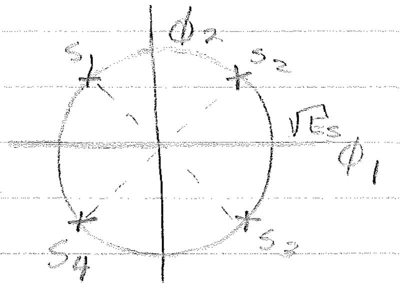


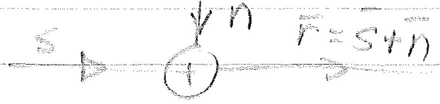
$a = \bar{\phi}_1 \cdot \bar{f} = \int_{-\infty}^{\infty} \phi_1(t)\, f(t)\, dt$

$b = \bar{\phi}_2 \cdot \bar{f} = \int_{-\infty}^{\infty} \phi_2(t)\, f(t)\, dt$

4 - 71

4-2)





$\bar{n} = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$, $p_n(n_1, n_2) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{\frac{-(n_1^2 + n_2^2)}{2\sigma^2}}$

$p_n(n_1 n_2) = \frac{1}{N_0 \pi} \cdot e^{-\frac{(n_1^2 + n_2^2)}{N_0}}$

$= \frac{1}{\sqrt{N_0 \pi}} e^{-\frac{n_1^2}{N_0}} \frac{1}{\sqrt{N_0 \pi}} e^{\frac{-n_2^2}{N_0}}$

$= p_n(n_1)\, p_n(n_2)$

WE CAN DEFINE 2 ORTHONORMAL TIME FUNCTIONS, $\phi_1(t), \phi_2(t)$.

THEN THE SIGNAL'S MAY BE WRITTEN:

$$\begin{cases} s_0(t) = \sqrt{E_s}\left(\frac{1}{\sqrt{2}}\phi_1(t) + \frac{1}{\sqrt{2}}\phi_2(t)\right) \\ s_1(t) = \sqrt{E_s}\left(\frac{1}{\sqrt{2}}\phi_1(t) + \frac{1}{\sqrt{2}}\phi_2(t)\right) \\ s_2(t) = -s_0(t) \\ s_3(t) = -s_1(t) \end{cases}$$

$$s_0 = \begin{bmatrix} \sqrt{\frac{E_s}{2}} \\ \sqrt{\frac{E_s}{2}} \end{bmatrix}; s_1 = \begin{bmatrix} \sqrt{\frac{E_s}{2}} \\ \sqrt{\frac{E_s}{2}} \end{bmatrix}; s_2 \begin{bmatrix} \sqrt{\frac{E_s}{2}} \\ \sqrt{\frac{E_s}{2}} \end{bmatrix}; s_3 = \begin{bmatrix} \sqrt{\frac{E_s}{2}} \\ -\sqrt{\frac{E_s}{2}} \end{bmatrix}$$

(HOW DO WE MAKE $n(t)$ TO A VECTOR

$$\left.\begin{aligned} n_1 &= \int_{-\infty}^{\infty} n_1(t)\phi_1(t)\,dt \\ n_2 &= \int_{-\infty}^{\infty} n_2(t)\phi_2(t)\,dt \end{aligned}\right\} \text{PROJECTION}$$

(ϕ_1 & ϕ_2 ORTHONORMAL)

n_1 AND n_2 ARE GAUSSIAN R.V.)

$P[m_i] = P[m_j] \quad i, j = 0, 1, 2, 3$

MUST MAXIMIZE APOSTERORI PROBIBILITY:

$P[m_i]\,p_r(\bar{\rho}\,|\,s = s_i)$; i.e

$\hat{m}(\rho) = m_i$ WHEN

$P[m_i]\,p_r(\rho\,|\,s = s_i) > P[m_k]\,p_r(\rho\,|\,s = s_k)$ $i \neq k$

BECAUSE, $P[m_i] = 1/4$, WE HAVE MAXIMUM LIKLIHOOD SITUATION, AND THE DECISION FUNC. BECOMES:

$p_r(\rho\,|\,s = s_i) \qquad (\bar{n} = \bar{r} - \bar{s}_k)$

NOW $p_r(\bar{\rho}\,|\,s = \bar{s}_k) = p_n(\bar{\rho} - \bar{s}_k)$

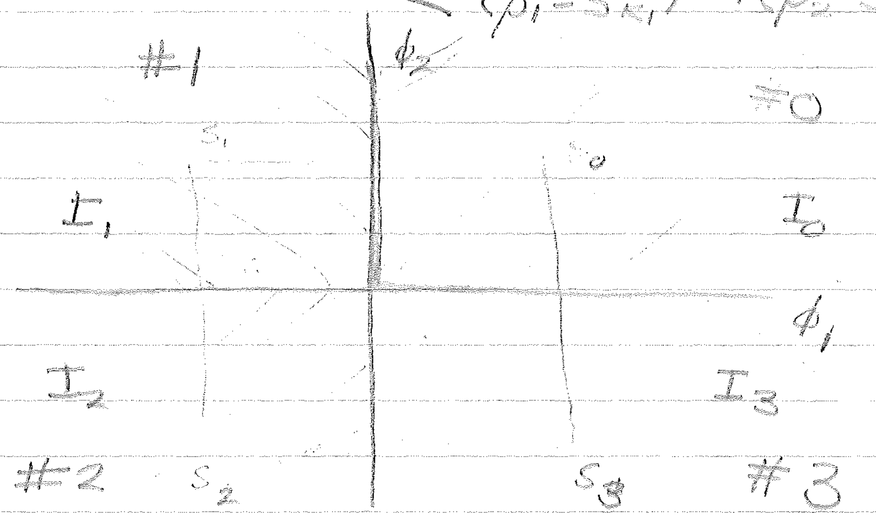
$$= \frac{1}{\pi N_0} e^{-[(\rho_1 - s_{k1})^2 + (\rho_2 - s_{k2})^2]/N_0}$$

AND $\frac{1}{2\pi N} e^{-[(\rho_1 - s_{i_1})^2 + (\rho_2 - s_{i_2})^2]/N_0}$

$> \frac{1}{\pi N} e^{-[(\rho_1 - s_{k1})^2 + (\rho_2 - s_{k_2})^2/N_0}$

IFF $(\rho_1 - s_{i_1})^2 + (\rho_2 - s_{i_2})^2$

$< (\rho_1 - s_{k_1})^2 + (\rho_2 - s_{k_2})^2$



$p[C/m_i] = p[r \in I_i / m_i]$

$= \iint_{I_i} p_r(\rho / s = s_i)\, d\rho$

FOR EXAMPLE:

$p[C/m_0] = \int_0^\infty \int_0^\infty \frac{1}{\pi N_0} e^{-[(\rho_1 - s_{01})^2 + (\rho_2 - s_{02})^2]/N_0}\, d\rho_1\, d\rho_2$

$= \int_0^\infty \int \frac{1}{\pi N_0} e^{-[(\rho_1 - \sqrt{\frac{E}{2}})^2 + (\rho_2 - \sqrt{\frac{E}{2}})^2/N_0}\, d\rho_1 d\rho_2$

$= \left(\left(\frac{1}{\pi N_0}\right)^{\frac{1}{2}} \int_0^\infty e^{-\frac{(\rho_1 - \sqrt{E/2})}{N_0}}\, d\rho_1\right.$

$\left(\frac{1}{\pi N_0}\right)^{\frac{1}{2}} \int_0^\infty e^{-\frac{\rho_2 - \sqrt{\ }}{N_0}}\Big)\, d\rho_2$

$= \left[\left(\frac{1}{\pi N_0}\right)^{\frac{1}{2}} \int_0^\infty e^{-\frac{\rho - \sqrt{\ }}{N_0}}\right)\, d\rho$

$= Q^2\left(-\sqrt{\frac{E}{N_0}}\right)$

$\Rightarrow Q(-\sqrt{\ }) = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{\ }}^\infty e^{-\beta^2/2}\, d\beta$

$\beta^2/2 = (\rho - \sqrt{\ })^2 \sqrt{2/N_0}$

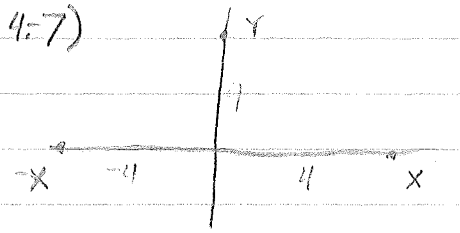
$\Rightarrow P[c/m_0] = Q^2\left(-\sqrt{\frac{E_s}{N_0}}\right)$

$\therefore P[c/m_i] = \quad ''$ $\quad \ni i = 0, 1, 2, 3$

$\Rightarrow P[c] = Q^2\left(-\sqrt{\frac{E_s}{N_0}}\right)$

$\frac{E_s}{N_0}$ = SIGNAL TO NOISE RATIO

4.7)



X AND -X BETTER THAN X AND Y (FURTHER APART)

DEFINE AN ORTHONORMAL SET

$\phi_1(t) = \frac{1}{4} x(t)$

$\phi_2(t) = \frac{1}{4} y(t)$

THEN $\int_{-\infty}^{\infty} \phi_1(t)\phi_2(t)dt = 0$

$\int_{-\infty}^{\infty} \phi_i^2(t)dt = 1 \qquad i = 1, 2$

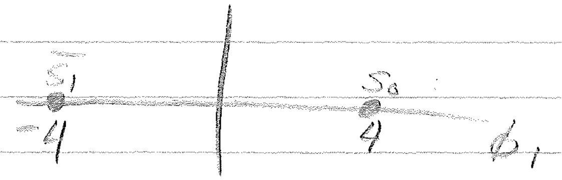
SPECTRAL DENSITY OF NOISE:

$S(f) = \frac{N_0}{2} = 4 \ \frac{\text{WATT-SEC}}{\text{CYCLE}} = 4 \text{ joule}$

$\Rightarrow N_0 = 8$

a) LET $s_0(t) = 4\phi_1(t) \Rightarrow \overline{\overline{S_0}} = 4\phi_1$

$s_1(t) = -4\phi_1(t) \Rightarrow \overline{\overline{S_1}} = -4\phi_1$



$P[m_0] = P[m_1] = \frac{1}{2}$

$\therefore \hat{m}(\rho) = m_0$ IFF

$p_n(\rho - s_0) > p_n(\rho - s_1)$

IN THE FIRST PROBLEM,

ALL VECTORS ARE ONE DIMENSIONAL

BY DEFINITION;

$\tilde{n} = n = \int_{-\infty}^{\infty} n_w(t)\,\phi_1(t)\,dt$ (an)

$E[n] = \int_{-\infty}^{\infty} E[n_w(t)]\,\phi_1(t)\,dt = 0$

$n^2 = \int^{\infty} n_w(t_1)\,\phi_1(t_1)\,dt_1 \int_{\infty} n_w(t_2)\,\phi_1(t_2)\,dt_2$

$= \int\int_{\infty}^{\infty} n_w(t_1)\,n_w(t_2)\,\phi_1(t_1)\,\phi_1(t_2)\,dt_1\,dt_2$

$$E[n^2] = \int\int_{\infty}^{\infty} \underbrace{E[n_w(t_1)\,n_w(t_2)]}_{\text{AUTOCORRELATION OF WHITE NOISE}}\,\phi_1(t_1)\,\phi_2(t_2)\,dt_1\,dt_2$$

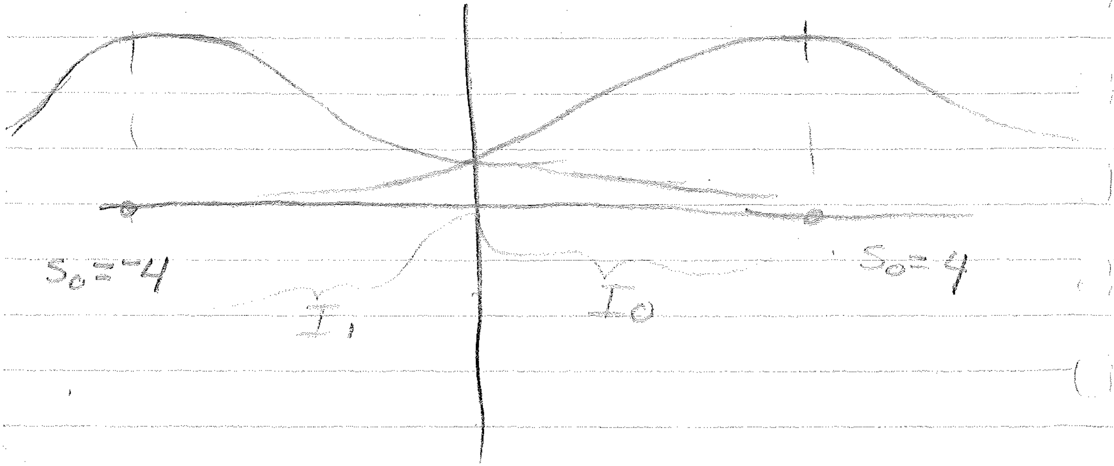
$$= \frac{N_0}{2}\int\int_{\infty}^{\infty} \delta(t_1 - t_2)\,\phi_1(t_1)\,\phi_1(t_2)\,dt_1\,dt_2$$

$$= \frac{N_0}{2}\int_{\infty}^{\infty} \phi_1(t_1)\,\phi_1(t_1)\,dt_1 = \frac{N_0}{2}$$

$E[n^2] = \sigma^2$ (FOR 0 MEAN) $= N_0/2$

$$\therefore p_n(\rho - s_0) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(\rho - s_0)^2}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{\pi N_0}}\, e^{-\frac{(\rho - s)^2}{N_0}} = \frac{1}{2(2\pi)^{1/2}}\, e^{-\frac{(\rho - s_1)^2}{8}}$$



$P[c/m_0] = \int_0^\infty \frac{1}{2\sqrt{2\pi}} e^{-\frac{(\rho-S_0)^2}{8}} d\rho$

LET $\mathcal{B} = \frac{\rho - S_0}{2}$; $d\mathcal{B} = \frac{1}{2} d\rho$

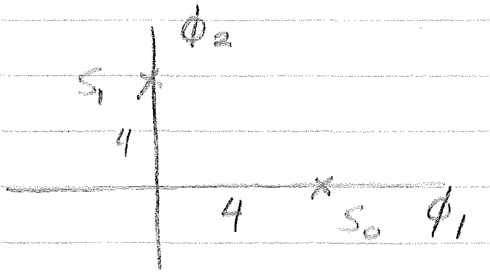
$\Rightarrow P[c/m_0] = \frac{1}{\sqrt{2\pi}} \int_{-2}^{\infty} e^{-\mathcal{B}^2/2} d\mathcal{B}$

$= Q(-2)$

$P[c/m_1] = P[c/m_0] = Q(-2)$

$\Rightarrow P[c] = Q(-2)$

b)



$s_0 = [4,0]$; $s_1 = [0,4]$

$\bar{n} = [n_1, n_2]$

$n_1 = \int_{-\infty}^{\infty} n_w(t)\phi_1(t)dt$

$n_2 = \int_{-\infty}^{\infty} n_w(t)\phi_2(t)dt$

$E[n_1] = E[n_2] = 0$

$E[n_1^2] = E[n_2^2] = N_0/2$

$E[n_1 n_2] = \iint E[n_w(t_1)n_w(t_2)]\phi_1(t_1)\phi_2(t_2)dt_1dt_2$

$= \frac{N_0}{2} \int^{\infty} \phi_1(t_1)\phi_2(t_1)dt_1 = 0$

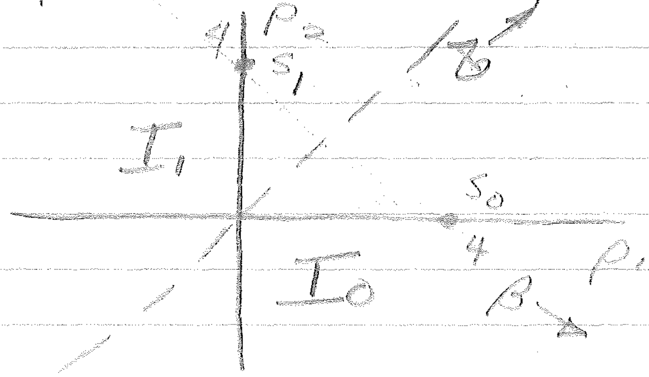
∴ NOISE COMPONENTS INDEPENDENT FOR THE PROJECTIONS ONTO TWO ORTHOGONAL FUNC, $\phi_1(t)$, $\phi_2(t)$

$\therefore \rho_n(\alpha) = \frac{1}{2\pi \frac{N_0}{2}} e^{-(\alpha_1^2+\alpha_2^2)/2 N_0/2}$

(CONT)

$$p_n(\rho - s_0) = \frac{1}{2\pi \frac{N_0}{2}} e^{-\left[\frac{(\rho_1-4)^2+\rho_2^2}{2N_0/2}\right]}$$

$$p_n(\rho - s_1) = \frac{1}{\pi N_0} e^{-\left[(\rho_1^2+(\rho_2-4)^2)/N_0\right]}$$



$$P[C/m_0] = \int_{-\infty}^{\infty} d\rho_2 \int_{\rho_2}^{\infty} d\rho_1 \frac{1}{\pi N_0} e^{-[(\rho_1-4)^2+\rho_2^2]/N_0}$$

LET $b = \rho_1 - 4$

$$P[C/m_0] = \int_{-\infty}^{\infty} d\rho_2 \int_{\rho_2-4}^{\infty} db \frac{1}{\pi N_0} e^{-\frac{(b^2+\rho_2^2)}{N_0}}$$

LET $\alpha = \frac{1}{\sqrt{2}}(b - \rho_2)$

$\beta = \frac{1}{\sqrt{2}}(b + \rho_2)$

THEN

$$P[C/m_0] = \int_{\infty}^{-\infty} d\beta \int_{-2\sqrt{2}}^{\infty} d\alpha \frac{1}{\pi N_0} e^{-(\alpha^2+\beta^2)/N_0}$$

$$= \frac{1}{\sqrt{\pi N_0}} \int_{-\infty}^{\infty} d\alpha\, e^{-\frac{\alpha^2}{N_0}} = Q(-\sqrt{2})$$

$P[C/m_1] = Q(-\sqrt{2})$

$\Rightarrow P[C] = Q[-\sqrt{2}] < Q[-2]$

5-13-71

RECEIVER IMPLEMENTATION

M-MESSAGE, $i = 0, \ldots, M-1$

N-DIMENSIONS $j = 1, \ldots, N$

RECEIVER CALCULATES:

$\bar{r} = (r_1, \ldots, r_2)$

$r_j = \int_{-\infty}^{\infty} r(t)\,\phi_j(t)\,dt$

= PROJECTION OF $r(t)$ ONTO $\phi_j(t)$

DECISION FUNCTION

$|\bar{r}-\bar{s}_i|^2 - N_0 \ln P[m_i]$

FOR W.G.N, $S_n f = \frac{N_0}{2}$; $\sigma^2 = \frac{N_0}{2}$

$\hat{m} = m_k$ IF DECISION FUNCTION

IS MINIMUM FOR $i = k$

NOW $|r-s_i|^2 = \sum_{j=1}^{N} (r_j - s_{ij})^2 = |\bar{r}|^2 - 2\bar{r}\cdot\bar{s}_i + |\bar{s}_i|^2$

$\Rightarrow \bar{r}\cdot\bar{s}_i = \sum_{j=1}^{N} r_j s_{ij}$

HENCE, BECAUSE OF THE (−) SIGN

BEFORE $\bar{r}\cdot\bar{s}_i$, MUST MAXIMIZE

$\bar{r}\cdot\bar{s}_i$ IN ORDER TO MINIMIZE

$|\bar{r}-\bar{s}_i|^2$

REALLY, WE MUST MAXIMIZE:

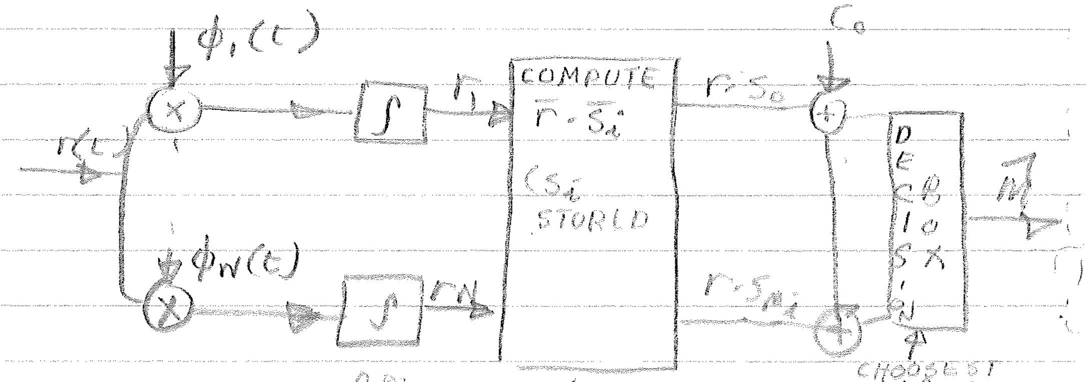
$\bar{r}\cdot\bar{s}_i + C_i \ni C_i = \frac{1}{2}(N_0 \ln P[m_i] - |\bar{s}_i|^2)$

$i = 0, \ldots, M-1$

(25)

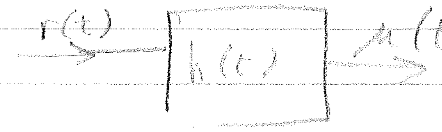
CORRELATION RECEIVER COMPUTES
THE (CROSS)CORRELATION
FUNCTION $\bar{r} \cdot \bar{S}_i$, EITHER
ANALOGICALLY OR DIGITLY

NOTE $\bar{r} \cdot \bar{S}_i = \int_{-\infty}^{\infty} r(t) s_i(t) dt$
OR $r_i = \int_{-\infty}^{\infty} r(t) \phi_i(t) dt$



$s_{ij} = \int_{-\infty}^{\infty} s_i(t) \phi_j(t) dt$

MATCHED FILTERS (Pg 236)



$\mu(t) = \int_{-\infty}^{\infty} r(\alpha) h(t-\alpha) d\alpha$

WE WANT TO GENERATE
THE COMPONENTS, r_i OF THE FORM
$r_i = \int_{-\infty}^{\infty} r(\alpha) \phi_i(\alpha) d\alpha$
CONSIDER;
$h(t) = \phi_i(T-t)$; THEN

$\mu(t) = \int_{-\infty}^{\infty} r\alpha(T-(t-\alpha)) d\alpha$

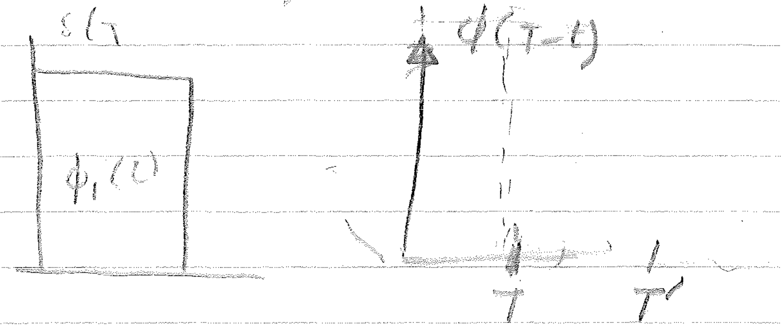
$= \int_{-\infty}^{\infty} r(\alpha)\phi_i(T+\alpha-t)\,d\alpha$

SAMPLE AT $t=0$ AND GET

$\mu_i(t) = \mu_i = \int_{-\infty}^{\infty} r(\alpha)(\phi_i(\alpha))d\alpha$

h(t) MUST BECAUSE

$\therefore h(-t)=0$; SUPER IMPULSE



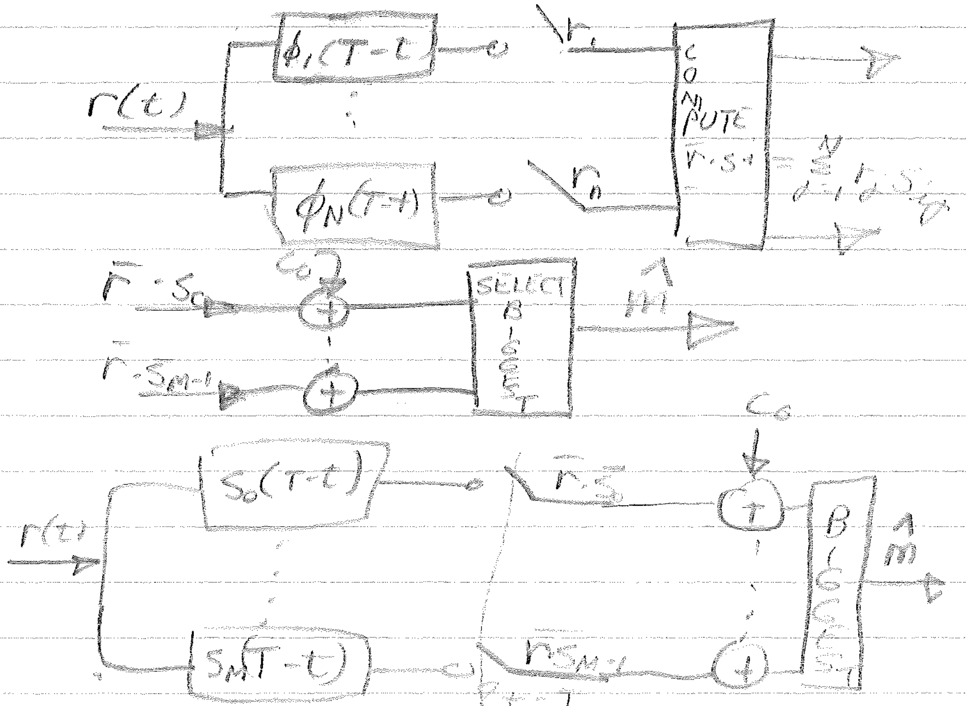
FOR CAUSALITY ON $\phi(T-t)$, WE FORGET ~~THE A~~, WE FORGIVE THAT EACH BASIC FUNC. VANISH IDENTICALLY OUTSIDE SOME FINITE INTERVAL $0 \le t \le T$

CAN MATCH DIRECTLY TO $S_i(t)$

$$u(t) = \int_{-\infty}^{\infty} r(\alpha) S_i(T-(t-\alpha))\, d\alpha$$

$$= \int_{-\infty}^{\infty} r(\alpha) S_i(T+\alpha-t)\, d\alpha$$

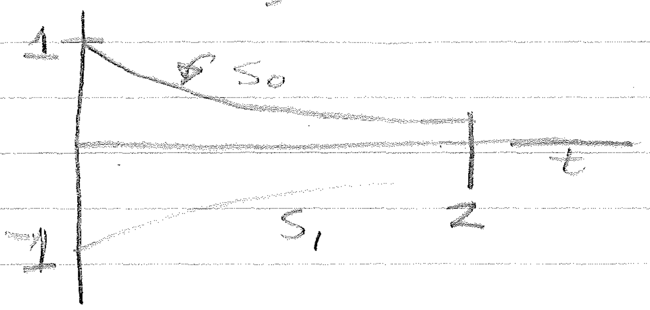
$$\therefore\ u(T) = \bar{r}\cdot\bar{S}_i = \int_{-\infty}^{\infty} r(\alpha) S_i(\alpha)\, d\alpha$$



MATCHING DIRECTLY TO SIGNAL

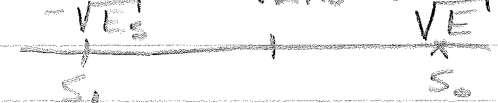
4-15)

$s_0 = e^{-t}\mu(t)\ ;\ s_1(t) = -e^{-t}\mu(t)$



ASSUMING $S_N(f) = {}^{N_0}/_2$

$P[E] = Q\left[\frac{d}{\sqrt{2N_0}}\right] \ni d =$ DISTANCE BETWEEN SIGNALS



$E_s = \int_0^{\infty} e^{-2t}\,dt = \frac{1}{2}$

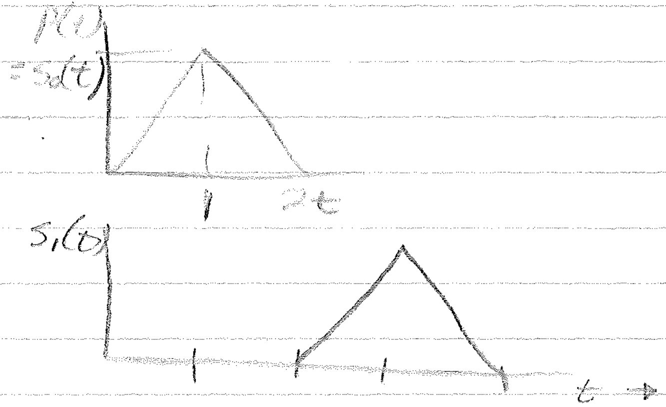
$\Rightarrow d = 2\sqrt{E_s}$

$\therefore\ P[E] = Q\left[\frac{2\sqrt{E_s}}{\sqrt{2N_0}}\right] = Q\left[\sqrt{\frac{2E_s}{N_0}}\right]$

E_s (REALLY) $= \int_0^2 e^{-2t}\,dt = \frac{1}{2}(1-e^{-4})$

(29)

4-10)



a) AN OPTIMUM RECEIVER
SELECTS m_0 AFTER
RECEIVING $r(t)$

$\int_{-\infty}^{\infty} r(t)s_0(t)dt > \int_{-\infty}^{\infty} r(t)s_1(t)dt$

SINCE THE ENERGY OF
$s_0(t)$ = ENERGY IN $s_1(t)$
AND BOTH ARE EQUALLY
PROBABLE
($C_0 = C_1$, HENCE, IGNORE)

b) FOR EQUALLY LIKELY BINARY
MESSAGES:

$$P[\varepsilon] = Q\left[\frac{d}{\sqrt{2N_0}}\right]$$

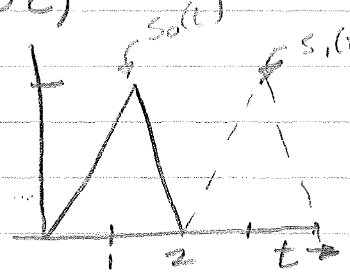
WHERE $d^2 = |s_0 - s_1|^2$

$$= \int_{-\infty}^{\infty} [s_0(t) - s_1(t)]^2 dt = \frac{16}{3}$$

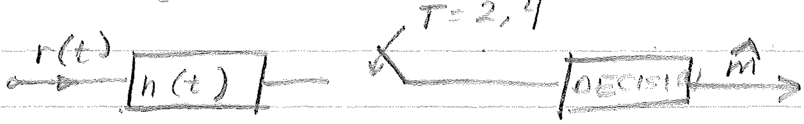
$$\Rightarrow P[E] = Q\left[\frac{4}{\sqrt{6(.1)}}\right] \sim 10^{-7}$$

4-17-71

4-10 c)



c) METHOD 1

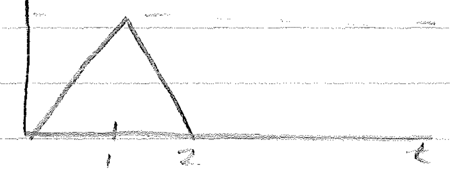


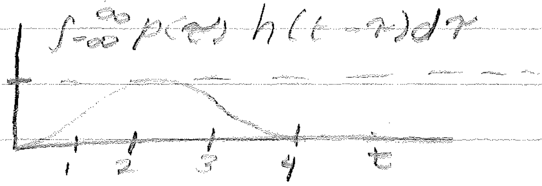
THE SAMPLE AT $T=2$ IS

$\int_{-\infty}^{\infty} r(t)s_0(t)dt,$

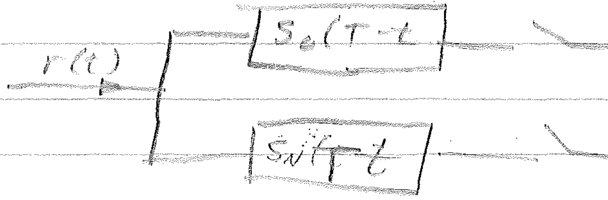
THAT AT $T=4$ IS $\int_{-\infty}^{\infty} r(t)s_1(t)dt$

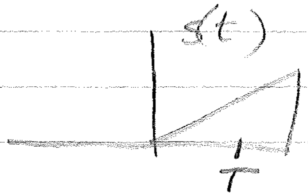
$h(t) = s_0(2-t)$

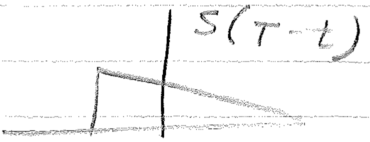




MATCH FILTER



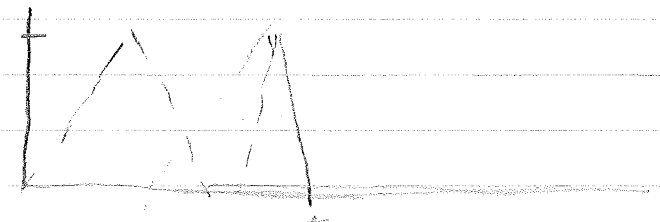




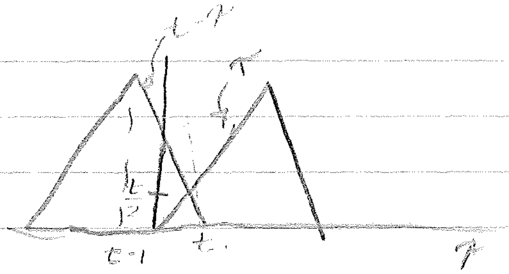
(CONT)

$$\int_{-\infty}^{\infty} r(\alpha)\, h(t-\alpha)\, d\alpha = v(t)$$
$$= \int_{-\infty}^{\infty} r(\alpha)\, s_0\left(T-(t-\alpha)\right) d\alpha$$
$$= \int_{-\infty}^{\infty} r(\alpha)\, s_0(T-t+\alpha)\, d\alpha$$
$$v(2) = \int_{-\infty}^{\infty} r(\alpha)\, s_0(\alpha)\, d\alpha$$
$$v(4) = \int_{-\infty}^{\infty} r(\alpha)\, s_0(\alpha-2)\, d\alpha$$
$$= \int_{-\infty}^{\infty} r(\alpha)\, s_1(\alpha)\, d\alpha$$

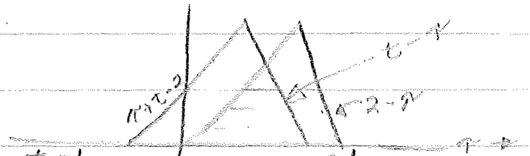
$$\int_{-\infty}^{\infty} p(\tau)\, s_0(t-\tau)\, d\tau$$
$$= \int_{-\infty}^{\infty} p(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} p(\tau)\, s_0(2-t+\tau)\, d\tau$$
$$= \int_{-\infty}^{\infty} s_0(\tau)\, s_0(2-t+\tau)\, d\tau$$
$$= \int_{-\infty}^{\infty} s_0(T)\, s_0(t-\tau)\, d\tau$$



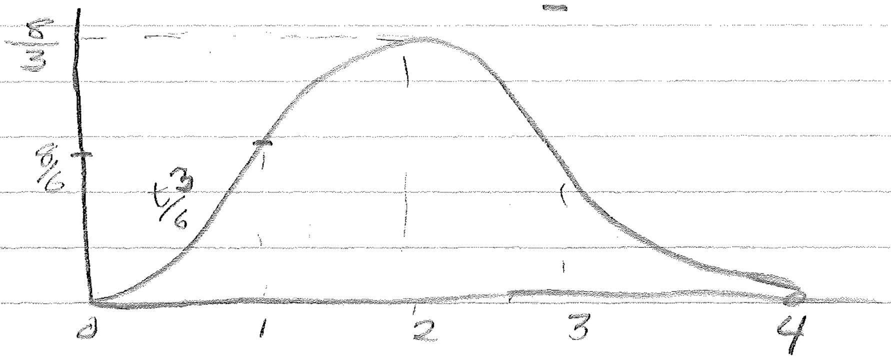
$t < 0 \Rightarrow * = 0$



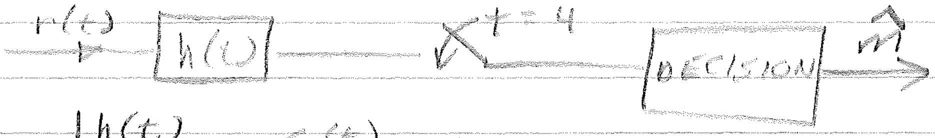
$$\int_0^t (t-\tau)\tau\, d\tau = -\frac{t^3}{3}+\frac{t^3}{2}=\frac{t^3}{6} \qquad 0<t<1$$

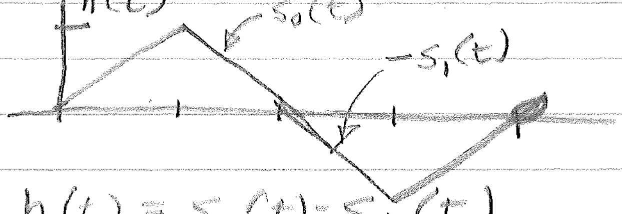


$$\int_0^{t-1}\tau(\tau+t-2) + \int_{t-1}^{1}\tau(t-\tau)d\tau + \int_1^t (t-\tau)(2-\tau)d\tau$$
$$=$$



FOR METHOD II





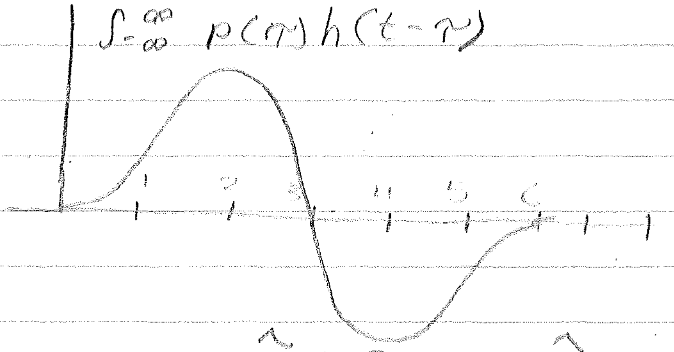
$$h(t) = s_0(t) - s_1(t)$$

(cont)

$\int_{-\infty}^{\infty} r(t)h(\tau-\tau)dt$

$= \int_{-\infty}^{\infty} r(t)\,[s_0(t)-s_1(\tau)]\,dt$

$\int_{-\infty}^{\infty} p(\tau)h(t-\tau)$



WRONG

FOR +, $\hat{m} = s_0$ $\hat{m} = s_1$ FOR −

$h(t) = s_1(4-t) - s_0(4-t)$

$\int_{-\infty}^{\infty} r(\tau)[s_1(4+\tau+t - s_0(4+\tau-t)d\tau)$

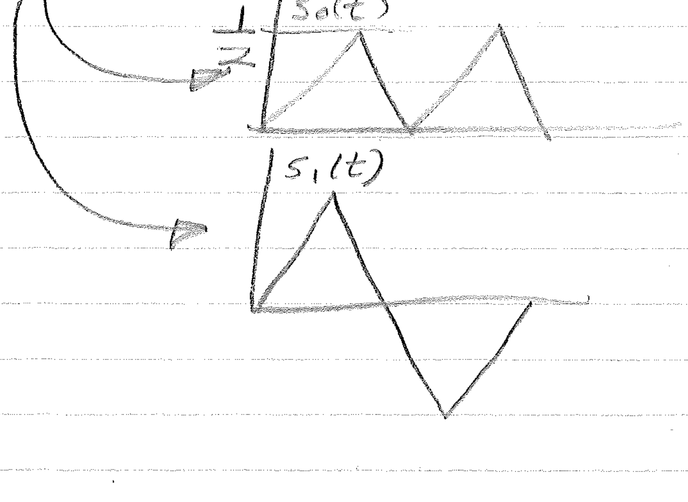
(RIGHT GRAPH)

d) SAME ERROR PROB.

$\{ s_0(t) = \frac{1}{2}[p(t)+p(t-2)]$

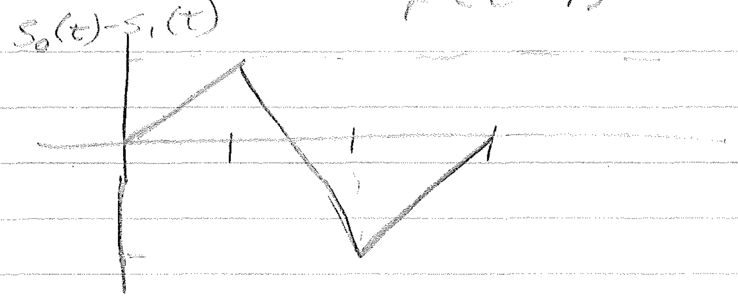
$\{$ SMALL ERROR (ENTIPITAL

$s_1(t) = p(t)$; $s_0(t) = p(t)$



e) FOR $S_0(t) = p(t)$

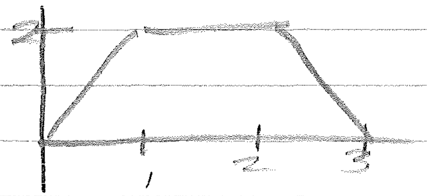
$S_1(t) = p(t-1)$



$$d^2 = \int_{-\infty}^{\infty}[S_0(t) - S_1(t)]^2 dt$$

$$= 2\int_0^1 (2x)^2 dx + 2\int_0^{\frac{1}{2}} (4x)^2 dx = 4$$

$$P[\varepsilon] = Q\left[\frac{1}{\sqrt{.05}}\right]$$



FOR $S_0(t) = p(t)$; $S_1(t) = p(t-1)$

$$d^2 = \int_{-\infty}^{\infty}[S_0(t) - S_1(t)]^2 dt = \frac{20}{3}$$

$$P[\varepsilon] = Q\left[\sqrt{\frac{5}{3(.05)}}\right] \cong 4 \times 10^{-9}$$

4-21-71

$\int_{-\infty}^{\infty} f(t)\, g(t)\, dt = \int_{-\infty}^{\infty} F(f)\, G(f)\, df$

$d^2 = \int_{-\infty}^{\infty} [s_1(t) - s_0(t)]^2 dt$

$= \int_{-\infty}^{\infty}$ [illegible] df

FOR REAL TIME FUNCTIONS;

$S_0(-f) = S_0^*(f)$

CHAPT. 7 : FIRST 2 SECTIONS

COMPLEX CONVOLUTION

$\mathcal{F}\{g(t)h(t)\} = \int_{-\infty}^{\infty} G(\beta) H^*(f-\beta)\, d\beta$

$G(f) = \mathcal{F}(g(t))$

$H(f) = \mathcal{F}(h(t))$

$\mathcal{F}(g(t)h(t)) = \int_{-\infty}^{\infty} e^{-j2\pi ft} g(t)h(t)\, dt$

$= \int_{-\infty}^{\infty} e^{-j2\pi ft} \left(\int_{-\infty}^{\infty} G(f_1) e^{j2\pi f_1 t} df_1\right) \int_{-\infty}^{\infty} H(f_2) e^{j2\pi f_2 t}$

$= \iint_{-\infty}^{\infty} G(f_1) H(f_2) \left\{\int_{-\infty}^{\infty} e^{-j2\pi(f-f_1-f_2)t} dt\right\} df_1\, df_2$

$\int_{-\infty}^{\infty} e^{-j\pi ft} dt \overset{\Delta}{=} \delta(f)$

$\Rightarrow \mathcal{F}(g(t)h(t)) = \iint H(f_2) \cdot G(f_1)\, \delta(f - f_1 - f_2)\, df_1\, df_2$

$= \int_{-\infty}^{\infty} G(\beta) H(f-\beta)\, d\beta$

FRI 4-25-71

7-4) $n(t) = n_c(t)\sqrt{2}\cos\omega_0 t + n_s(t)\sqrt{2}\sin\omega_0 t$

$$n(t)n(t-\tau) = [n_c(t)\sqrt{2}\cos\omega_0 t + n_s(t)$$
$$\sqrt{2}\sin\omega_0 t\,][n_d(t-\tau)\sqrt{2}\cos\omega_0(t-\tau)$$
$$+ n_s(t-\tau)\sqrt{2}\sin\omega_0(t-\tau\,]$$
$$= n_c(t)\,n_d(t-\tau)\,2\cos\omega_0 t\,\cos\omega_0 t-\tau$$
$$+ n_s(t)\,n_s(t-\tau)\,2\sin\omega_0 t\,\sin\omega_0(t-\tau$$
$$+ n_s(t)\,n_c(t-\tau)\,2\sin\omega_0 t\,\cos\omega_0(t-\tau$$
$$+ n_c(t)\,n_s(t-\tau)\,2\cos\omega_0 t\,\sin\omega_0(t-$$
$$= R_c(\tau)\{\cos\omega_0 t + \cos(2\omega_0 t - \omega_0\tau))$$
$$+ R_s(\tau)[\cos\omega_0\tau - \cos(2\omega_0 t - \omega_0\tau)]$$
$$+ R_{sc}(\tau)[\sin(2\omega_0 t - \omega_0\tau) + \sin\omega_0\tau]$$
$$+ R_{cs}(\tau)[\sin(2\omega_0 t - \omega_0\tau) - \sin\omega_0\tau]$$

WANT 2 MAKE FUNCTION OF τ

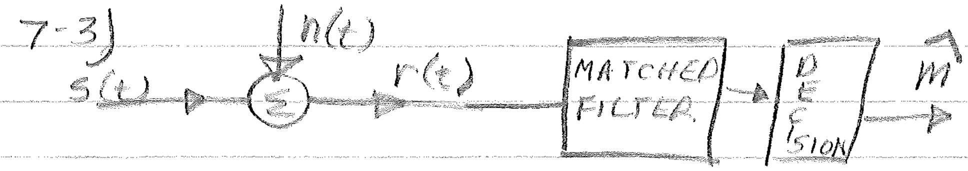
$$n(t)n(t-\tau) = \cos\omega_0\tau[R_c(\tau) + R_s(\tau)]$$
$$+ \cos(2\omega_0 t + \omega_0\tau)[R_c(\tau) - R_s(\tau)]$$
$$+ \sin\omega_0\tau\,[R_{sc}(\tau) - R_{cs}(\tau)]$$
$$+ \sin(2\omega_0 t - \omega_0\tau)[R_{sc}(\tau) + R_{cs}(\tau)]$$

WIDE SENSE STATIONARY SUFFICIENT:

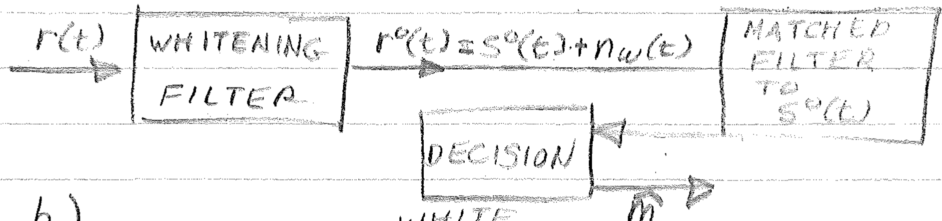
$$R_c(\tau) = R_s(\tau)$$
$$R_{sc}(\tau) = -R_{cs}(\tau)$$

(37)

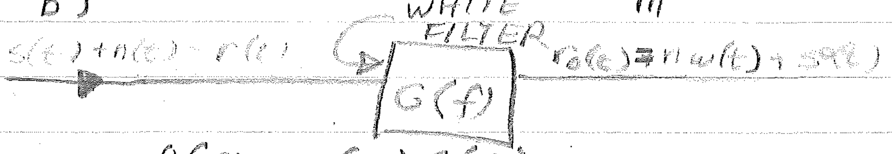
7-3)



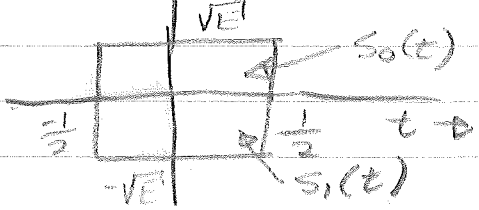
WISH TO PASS r(t) THRU WHITNING FILTER



b)



$$s^o(f) = G(f)S(f)$$



$$s^o_\bullet(f) = G(f)S(f)$$

(VOLTAGE)

WANNA FIND G(f)

$$S_w(f) = \frac{N_0}{2}$$

$$S_n(f) = \frac{N_0}{2} \frac{f^2+1}{f^2+4}$$

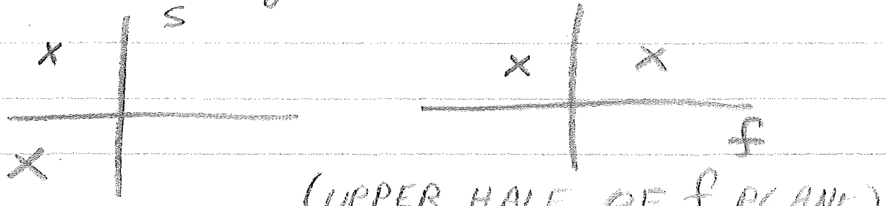
$$S_w(f) = |G(f)|^2 S_n(f)$$ (POWER)

$$\Rightarrow \frac{N_0}{2} = G(f)G^*(f) \frac{N_0}{2} \frac{f^2+1}{f^2+4}$$

$$\Rightarrow G(f)G^*(f) \frac{f^2+1}{f^2+4} = 1$$

$$\Rightarrow G(f)G^*(f) = \frac{1+jf}{2+jf} \frac{1-jf}{1-jf}$$

$$\therefore G(f) = \frac{2+jf}{1+jf}$$ (POLES IN LEFT)



(UPPER HALF OF f PLANE)

$$\left(\begin{array}{l} s = j\pi 2f \\ G(s) = \frac{4\pi + j2\pi f}{2\pi + j2\pi f} = \frac{4\pi + s}{2\pi + s} \end{array}\right)$$

$$\begin{array}{r|l} & 1 + \frac{2\pi}{s+2\pi} \\ s+2\pi & s+4\pi \\ & \underline{s+2\pi} \\ & 2\pi \end{array} \quad = 1 + \frac{2\pi}{s+2\pi}$$

$\Longrightarrow g(t) =$ IMPULSE RESPONSE FOR WHITENING FILTER

$$g(t) = \mathcal{L}^{-1}[G(s)] = \delta(t) + 2\pi e^{-2\pi t} \mu_{-1}(t$$

∴ OUTPUT SIGNAL FROM WHITENING FILTER (SIGNAL TO BE MATCHED)

$$s^o(t) = \int_{-\infty}^{\infty} s(\alpha) g(t-\alpha)\, d\alpha$$

$$= \pm\sqrt{E_s} \int_{-\frac{1}{2}}^{\frac{1}{2}} [\delta(t-\alpha) + e^{-(t-\alpha)} \mu_{-1}(t-\alpha)]\, d\alpha$$

FOR $t < -1/2$, $s^o_i(t) = 0$

FOR $-\frac{1}{2} < t < \frac{1}{2}$

$$s^o_i(t) = \pm\sqrt{E_s}\left(1 + e^{-t} \int_{-\frac{1}{2}}^{t} e^{\alpha} d\alpha\right)$$

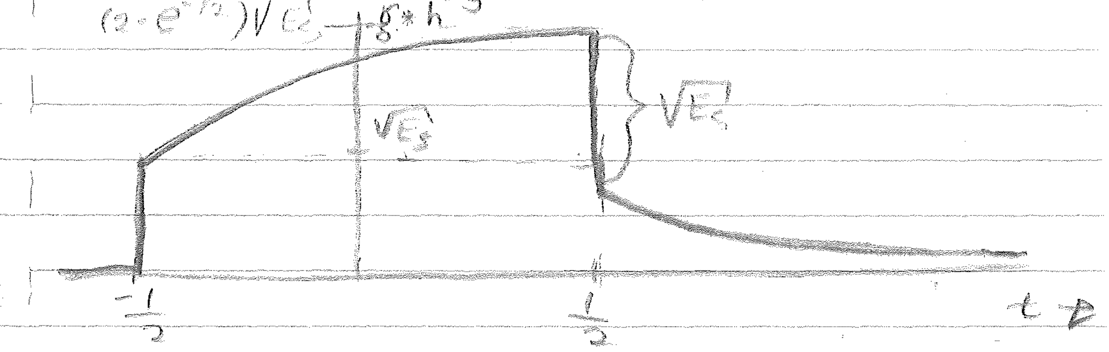
$$= \pm\sqrt{E_s}\left[2 - e^{-(t+\frac{1}{2})}\right]$$

FOR $t > \frac{1}{2}$

$$s^o(t) = \pm\sqrt{E_s}\left[e^{-t} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{\alpha} d\alpha\right]$$

$$= \pm\sqrt{E_s}\left(e^{1/2} - e^{-1/2}\right) e^{-t}$$

$$= \pm\sqrt{E_s}\, 2 \sinh\left(\tfrac{1}{2}\right) e^{-t}$$



THUS, WE CANNOT BUILD A MATCHED FILTER SAMPLING AT TIME T, CAUSE OF THE INFINATE TAIL,

CHOP OFF TAIL IN ABOUT 5 TIME CONSTANTS (5 sec), WITHOUT TO MUCH LOSS OF OPTIMALITY

MONDAY

7-5) $s_1(t)$, $s_2(t)$ - TWO LOW-PASS (BASE-BAND) SIGNALS

$s_1(t)\sqrt{2}\cos 2\pi f_0 t$

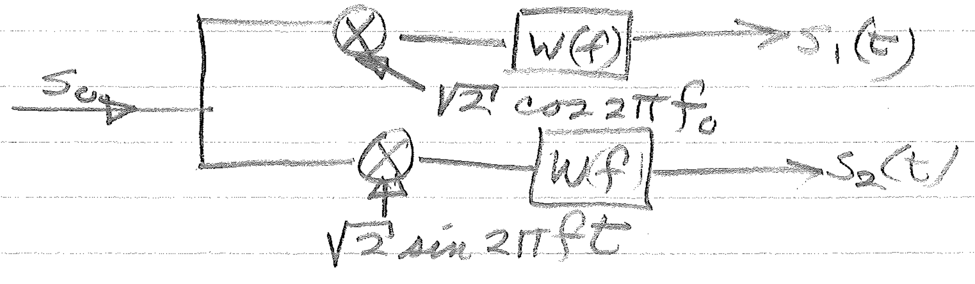
$s_2(t)\sqrt{2}\sin 2\pi f_0(t)$

$s_0(t)=s_1(t)\sqrt{2}\cos 2\pi f_0 t+s_2(t)\sqrt{2}\sin 2\pi f_0 t$

(QUADRITURE MULTIPLEXING)

IGNORE CHANNEL EFFECTS

HOW DO WE RECOVER $s_1(t)$ & $s_2(t)$



PROOF

SPECTRUM OF $s_0(t)$:

$$S_0(f) = \frac{1}{\sqrt{2}}\Big[S_1(f-f_0) + S_1(f+f_0) - jS_2(f-f_0) + jS_2(f+f_0)\Big]$$

$$\mathcal{F}\{s_1(t)\cos 2\pi f_0 t\}$$

$$= \mathcal{F}\{\tfrac{1}{2}s_1(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t})$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} S_1(\rho)\,\delta(f+f_0-\rho)\,d\rho$$

$$+ \frac{1}{2}\int_{-\infty}^{\infty} S_1(\rho)\,\delta(f+f_0-\rho)\,d\rho$$ (COMPLE. CONVOLUTIO

$$(\mathcal{F}(e^{j2\pi f_0(t)}) = \int_{-\infty}^{\infty} e^{-j2\pi(f-f_0)t}\,dt = \delta(f$$

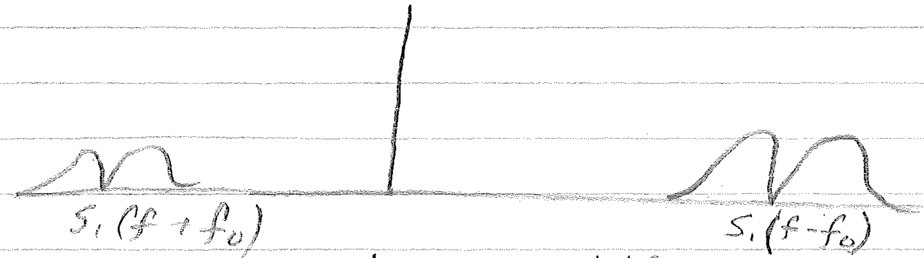
$$\Rightarrow \mathcal{F}\{s_1(t)\cos 2\pi f_0 t\}$$

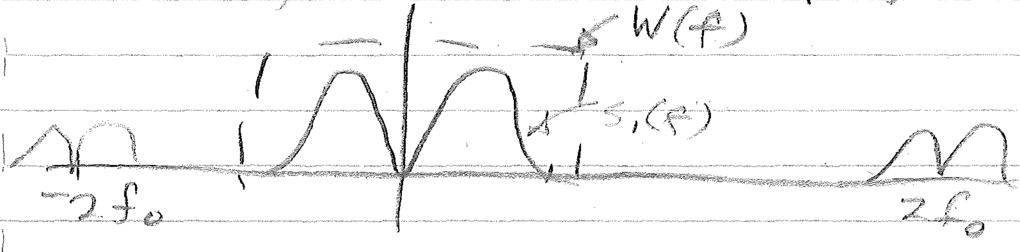
$$= \frac{1}{2}\Big(S_1(f-f_0) + S_1(f+f_0)\Big)$$

SHIFT TO RT

SHIFT LEFT

$$\therefore S_0(f) = \frac{1}{\sqrt{2}}\Big[S_1(f-f_0) + S_1(f+f_0) - jS_2(f-f_0) + jS_2(f+f_0)\Big]$$





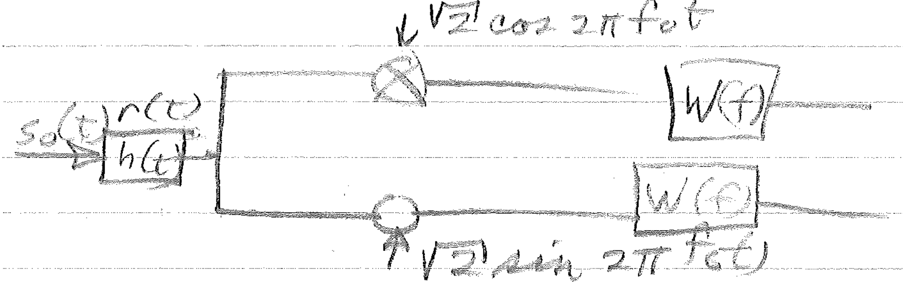
$-j S_2(f-2f_0) + \cancel{j S_2(f)}$
$\quad \cancel{- j S_2(f)} + j S_2(f+f_0)$

NOTHING AT f ORIGIN WHEN

MULTIPLIED BY $\cos 2\pi f_0 (t)$

NOW, IF CHANNEL'S IMPULSE

IS $R(f) = S_0(f) H(f)$



THE INPUT TO THE LOW-PASS FILTERS

COS CHANNEL

$\sqrt{2}\, r(t) \cos 2\pi f_0 t \rightarrow$ SPECT $\frac{1}{\sqrt{2}} [R(f-f_0)$
$\quad + R(f+f_0)]$

$= \frac{1}{\sqrt{2}} S_0(f-f_0) H(f+f_0) + (f+f_0) S_0$
$\quad H(f+f_0)$

$= \frac{1}{2} [H(f-f_0)[S_1(f-2f_0) + S_1(t)$
$\quad - j S_2(f-2f_0) + j S_2(t)]$
$\quad + H(f+f_0)[S_1(t) + S_1(f+2f_0)$
$\quad - j S_2(f) + j S_2(f+2f_0)]]$

SIN CHANNEL

$$\frac{1}{\sqrt{2}}\left[R(f-f_0)-R[f+f_0)\right]$$

$$=\frac{1}{\sqrt{2}}\left[S_0(f-f_0)H(f-f_0)\right.$$

$$\left.-S_0(f+f_0)H(f+f_0)\right]$$

$$=\frac{1}{\sqrt{2}}\{H(f-f_0)[S_1(f-2f_0)$$

$$+S_1(f)-j\,S_2(f-2f_0+j\,S_2(f)]$$

$$-H(f+f_0)[S_1(f)+S_1(f+2f_0)$$

$$-j\,S_1(f)+j\,S_2(f+2f_0)]\}$$

OUTPUT ONLY CONTAINS $S_n(f)$

$\Rightarrow$ OUTPUT OF COS CHANNEL

$$\frac{1}{2}\{S_1(f)\}[H(f-f_0)+H(f+f_0)]$$

$$+j\,S_2(f)[H(f-f_0)+H(f+f_0)]$$

$$0\leq f\leq W$$

$\Rightarrow$ OUTPUT OF SIN CHANNEL

$$\frac{1}{2}[S_1(f)][H_1(f-f_0)-H(f+f_0)]$$

$$+j\,S_2(f)[H(f-f_0)+H(f+f_0)]$$

$$0\leq f\leq W$$

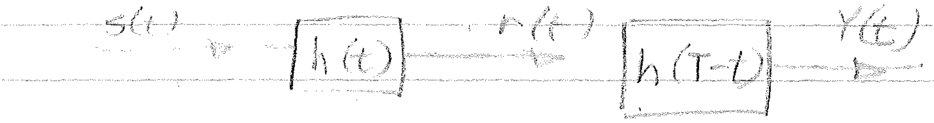
FOR $0\leq f\leq W,$

$$H(f-f_0)=H(f+f_0)$$

$$H(\omega)=H^*(-\omega)$$

$$\Rightarrow H^*(f_0-f)=H(f+f_0)$$

FRI.

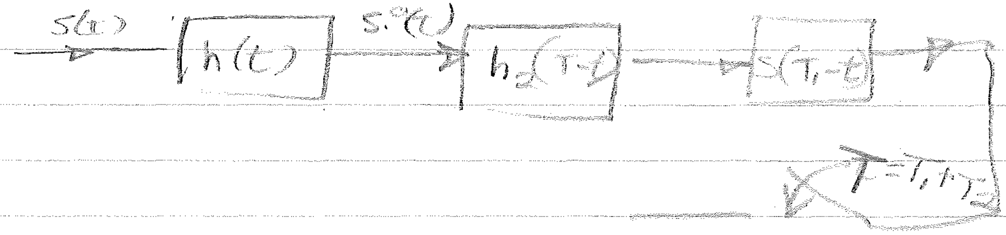


$$Y(t) = \int_{-\infty}^{\infty} r(\tau_1) h(T+\tau_1 - t)\, d\tau_1$$

$$R(f) = S(f) H(f)$$

$$Y(f) = R(f) H^*(f) e^{-j 2\pi f t}$$

$$= S(f) |H(f)|^2 e^{-j 2\pi f t}$$



SEQUENCE OF INPUT PARAMETERS COMMUNICATING THE FOLLOWING RANDOM VECTOR $\overline{m} = (m_1, m_2 \ldots m_k)$

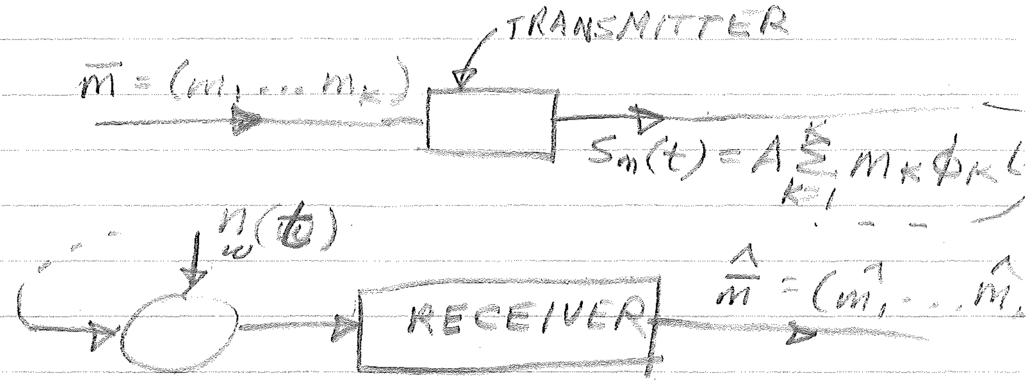
EACH m_i IS CONTINUOUS r.V.

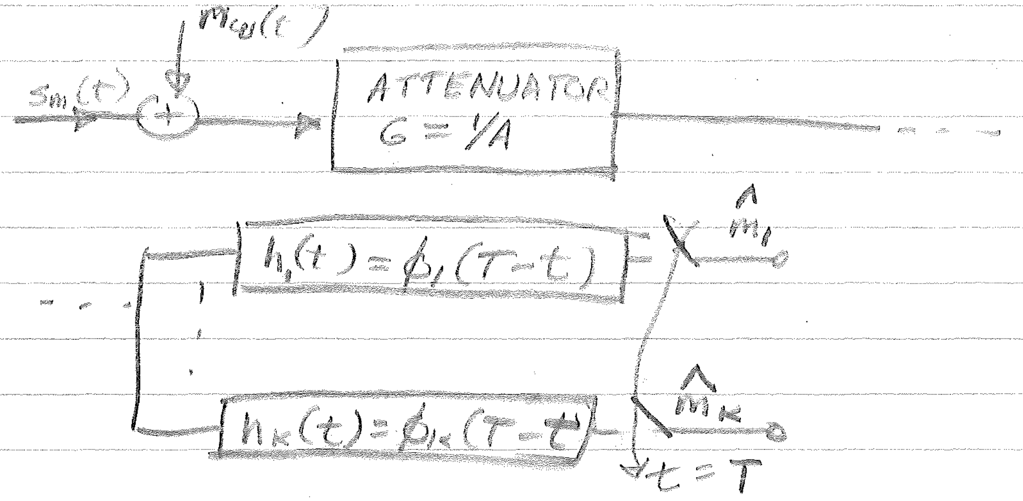
PRIOR TO THIS, ALL MESSAGES WERE DISCRETE OR FINITE INF

LINEAR MODULATION:

$$S_m(t) = \underset{\text{TRANSMITTOR GAIN}}{\uparrow A} \sum_{m=1}^{M} m_K \overset{\text{ORTHONORMAL}}{\phi_K}(t)$$

OBJECT: ESTIMATE $\hat{m} = (\hat{m}_1 \ldots \hat{m}_K)$





INPUT INTO MATCHED FILTER

$$\frac{r(t)}{A} = \sum m_K \phi_K(t) + \frac{n(w)}{A}$$

OUTPUT OF FILTER ①

$$= \sum_K m_K \int_{-\infty}^{\infty} \phi_K(t)\phi_1(T+\tau-t)\,d\tau$$
$$+ \frac{1}{A}\int_{-\infty}^{\infty} n_w(\tau)\phi_1(T+\tau-t)\,d\tau$$
$$= \sum m_K \int_{-\infty}^{\infty} \phi_K(\tau)\phi_1(\tau)\,d\tau$$
$$+ \frac{1}{A}\int_{-\infty}^{\infty} n_w(\tau)\phi_1(\tau)\,d\tau = \hat{m}_1$$
$$= m_1 + \frac{1}{A} n_1 \equiv \rho_1/A \ni \rho_1 = Am_1 = n_1$$

= RECEIVED 1ST COMPONENT

ERROR $= \hat{m}_1 - m_1 = \frac{1}{A} n_1$

MEAN SQUARE ERROR

PER COMPONENT

$= E[|\hat{m}_1 - m_1|^2] = E[\frac{1}{A^2} n_1^2]$

$= \frac{N_0}{2A^2} \Rightarrow N_0/2$ IS

SPECTRAL DENSITY OF

THE WHITE GAUSSIAN NOISE

$$\overline{\varepsilon^2} = \frac{1}{K} \sum_k E[|\hat{m}_k - m_k]^2]$$

$$= \frac{1}{K} E[(\hat{m}_1 - m_1)^2 + \ldots + (\hat{m}_K - m_K)^2]$$

$$= \frac{1}{K} E[|\hat{m} - \overline{m}|^2)$$

$$= \frac{1}{K} \frac{K N_0}{2A^2} = \frac{N_0}{2A^2}$$

(AGREES WITH ABOVE)

$$p_m(\alpha) p_r(\rho | m = \alpha)$$

$$= p_{r,m}(\rho, \alpha) = p_m(\underset{\uparrow}{\alpha | r = \rho}) p_r(\rho)$$

IN ORDER THAT THE A POSTERIORI PROB. DENSITY $p_m(\alpha | r = \rho)$ BE MAXIMIZED, WE ~~WA~~ MUST $\frac{p_m(\alpha) p_r(\rho | m = \alpha)}{p_r(\rho)}$ WITH RESPECT TO α

$$\Rightarrow p_m(\alpha)\underbrace{p_r(\rho/m)}_{\text{LIKELIHOOD FUNCTION}} \text{ IS MAX}$$

UPON MAXIMIZING, OUR LIKELIHOOD FUNCTION WE HAVE MAXIMUN LIKELIHOOD RECEPTION

$$r = n + m$$

$$\Rightarrow p_r(\rho / m=\alpha) = p_n(\rho-\alpha | m=\alpha)$$

NOISE & MESSAGE IND.

$\Longrightarrow = p_n(\rho-\alpha)$, WHICH CAN BE MAXIMIZED.

IN K DIMENSIONS

BECAUSE $n_w(t)$ IS WHITE GAUSSIAN NOISE, THE LIKELIHOOD FUNCTION IN K DIMENSIONS IS

$$p_r(\rho/m=\alpha) = p_n(\rho-\alpha A)$$

$$= \frac{1}{(\pi N_0)^{K/2}} e^{-|\rho-\alpha A|^2/N_0}$$

$$= \prod^{K} \frac{1}{\sqrt{\pi N_0}} e^{-(\rho_K - \alpha_K A)^2/N_0}$$

(NOTE $r_K = A m_K + n_K$)

THE VECTOR $\overline{\alpha}$ THAT MAXIMIZES THIS FUNCTION (FOR A GIVEN VALUE $\overline{\rho}$ OF THE RANDOM VECTOR $\overline{r}$) IS

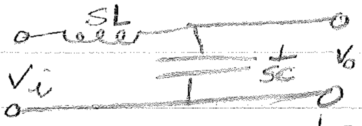
$$\overline{\alpha} = \overline{\rho}/A = \overline{m} + \overline{n}/A = \hat{\overline{m}}$$

THIS IS A LINEAR RECEIVER BECAUE THE ESTIMATE $\hat{\overline{m}}$ OF $\overline{m}$ IS A LINEAR FUNCTION OF THE RECEIVED VECTOR $\overline{\rho}$

RECALL $\overline{m} = \overline{m} + \overline{n}/A$ WAS ALSO THE RECEIVER IMPLEMENTED BEFORE

MONDAY

7-2 ex)



$$\frac{V_0}{V_i} = \frac{\frac{1}{SC}}{SL + \frac{1}{SC}} = \frac{1/LC}{S^2 + \omega_0^2}$$

SAMPLING THEOREM: OR HOW TO REDUCE CONTINUOUS (ANALOG) DATA TO DISCRETE (DIGITAL)

IF $z(t)$ IS A FINITE ENERGY WAVEFORM WHOSE FORIER TRANSFORM IS IDENTICALLY 0 FOR $|f| \geq W_m$ THEN

$$z(t) = \sum_{k=-\infty}^{\infty} z_k \psi_k(t)$$

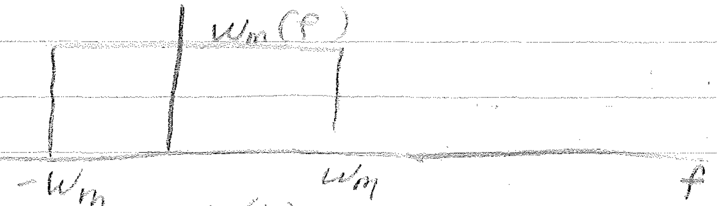
WHERE $z_k = \int_{-\infty}^{\infty} z(t)\, \psi_k(t)\, dt$

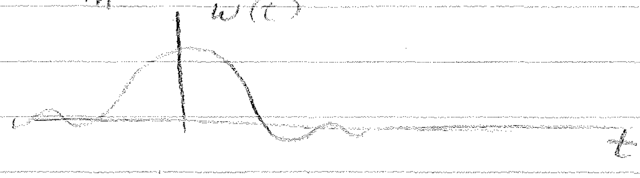
AND $\psi_k(t) = \psi\left(t - \frac{k}{2W_m}\right)$

$$\psi(t) = \sqrt{2W_m}\; \frac{\sin 2\pi W_m t}{2\pi W_m t}$$

OBERVATIONS:

① IMPULSE RESPONSE OF L.P. FILTER





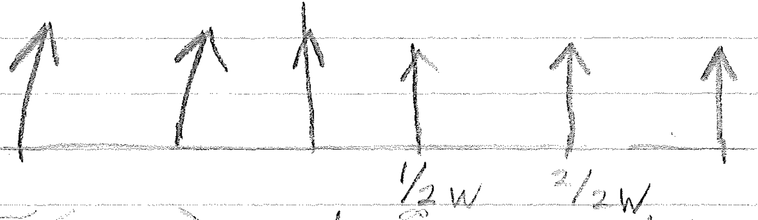
② $\{\psi_k(t)\}$ ARE ORTHONORMAL

$$\int_{-\infty}^{\infty} \psi_j(t)\, \psi_k(t)\, dt = \int_{-\infty}^{\infty} \psi_j(f)\, \psi_k(f)\, df$$
$$= 0 \qquad i \neq k$$
$$= 1 \qquad i = k$$
$$= \delta_{ij}$$

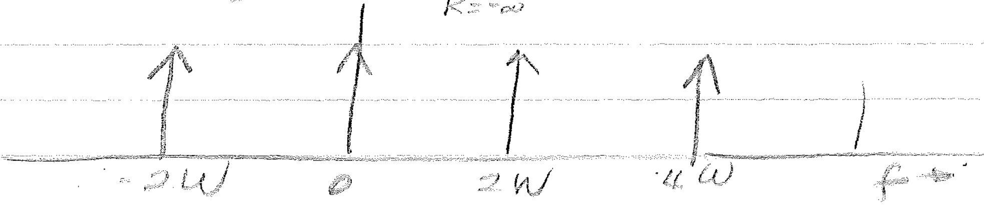
$$z(t)=\sum_{\ell=\infty}^{\infty} K_\ell \int_{-\infty}^{\infty} \psi_\ell(t)\,\psi_k(t)\,dt$$

IMPULSE MODULATION

$$u(t)=\sum_{k} \delta\left(t-\frac{k}{2W}\right)=\int_{-\infty}^{\infty} U(f)\,e^{j2\pi ft}\,df$$



$$\mathcal{F}(u(t)) = f\sum_{K=-\infty}^{\infty}\delta(f-2kW$$

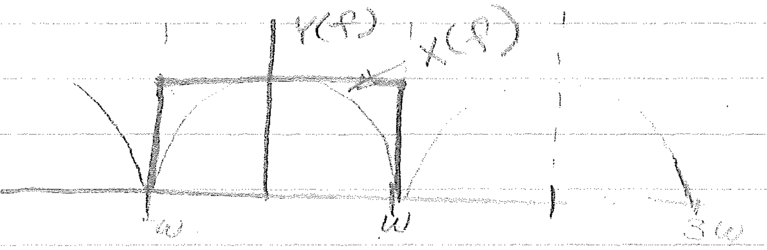


<u>OUTPUT OF IMPULSE MODULATOR</u>

$$y(t)=x(t)u(t)=\sum_{K=-\infty}^{\infty} x\left(\frac{k}{2W}\right)\delta(t-k/2W)$$

THEN

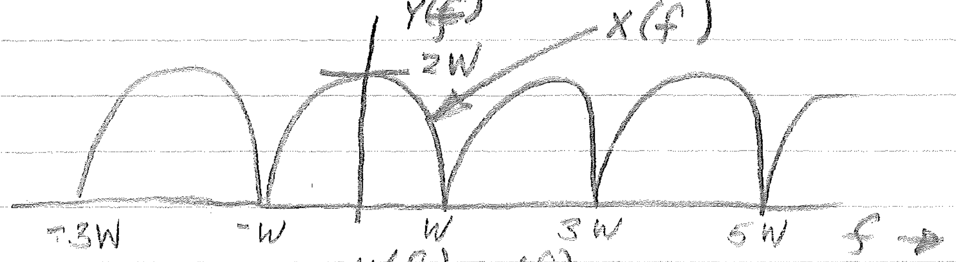
$$Y(f)=\int_{-\infty}^{\infty} X(\beta)\,U(f-\beta)\,d\beta$$
$$=2W\sum_{K,\infty}^{\infty} X(f-2kW)$$

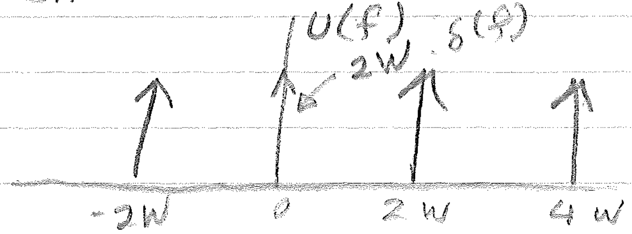


MON : TWO WEEKS FROM TODAY

IMPULSE TRAIN

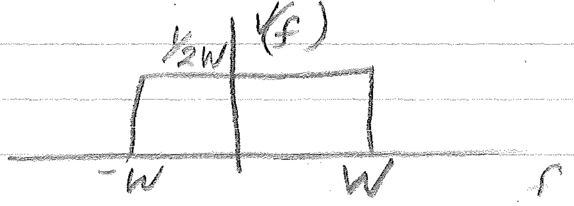
$$Y(t) = X(t)\,U(t) = \sum_{K=-\infty}^{\infty} X\left(\frac{K}{2W}\right)\delta\left(t-\frac{K}{2W}\right)$$





CAN REGAIN THRU FILTER

$\Rightarrow X(f) = Y(f)\ V(f)$



OR $\quad X(t) = Y(t) * V(t)$

$$= \int_{-\infty}^{\infty} Y(\tau)\,V(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} \sum_{K=-\infty}^{\infty} X\left(\frac{K}{2W}\right)\delta\left(t-\frac{K}{2W}\right) V(t-\tau)\,d\tau$$

$$= \sum_{K=-\infty}^{\infty} X\left(\frac{K}{2W}\right)\underbrace{V\left(t-\frac{K}{2W}\right)}_{\text{INTERPOLATION FUNKTION}}$$

$$= \sum_{K=-\infty}^{\infty} X\left(\frac{K}{2W}\right)\frac{\sin 2\pi W(t-K/2W)}{2\pi W(t-K/2W)}$$

FIG 8-13b

RANDOM WAVE FORM: $m(t)$ BANDLIMITED.

$\ni m(t) = \sum_{K=1}^{K} m_K \psi_K(t)$

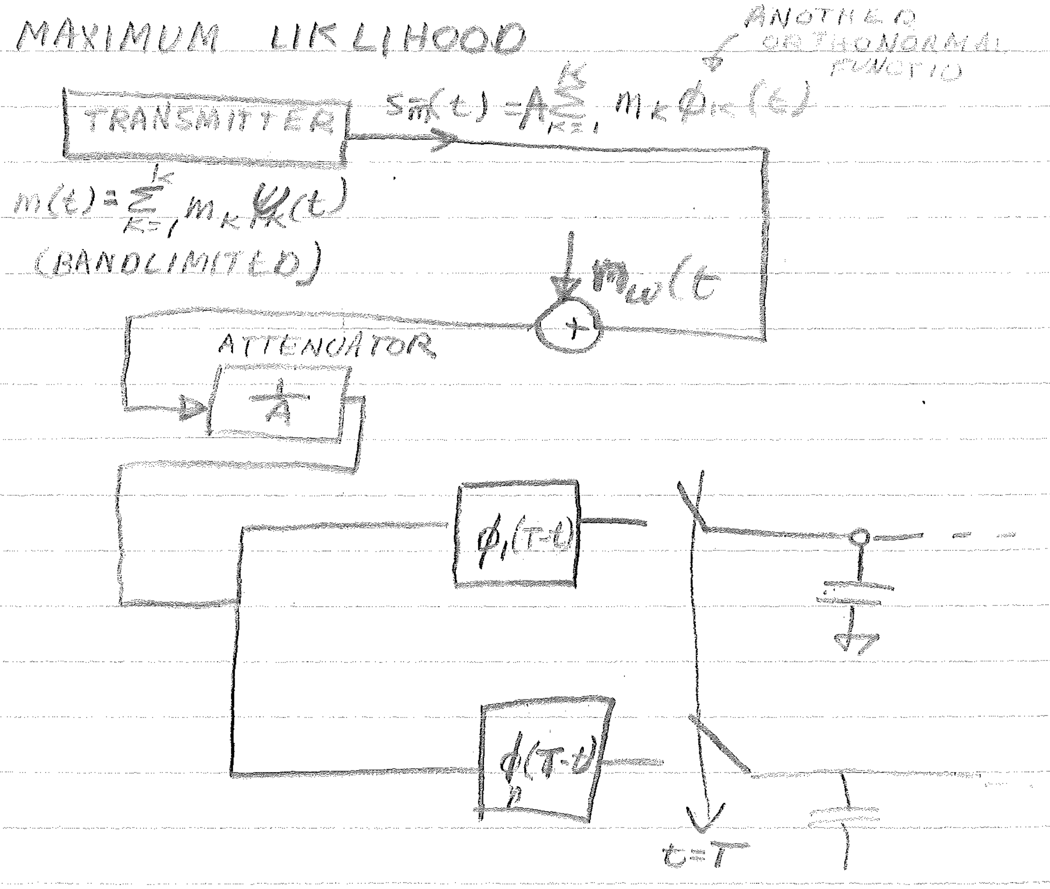
$\ni m_K = \int_{-\infty}^{\infty} m(t)\, \psi_n(t) = \frac{1}{\sqrt{2W_m}} m\left(\frac{K}{2W_m}\right)$

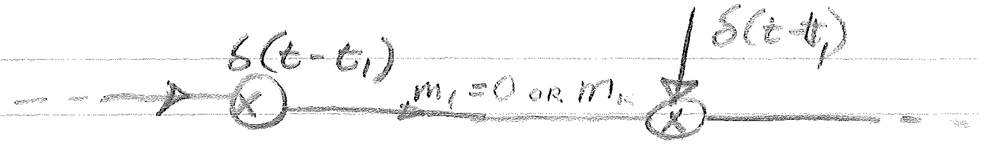
COMMUNICATE, AND THEN ESTIMATE THIS DATA

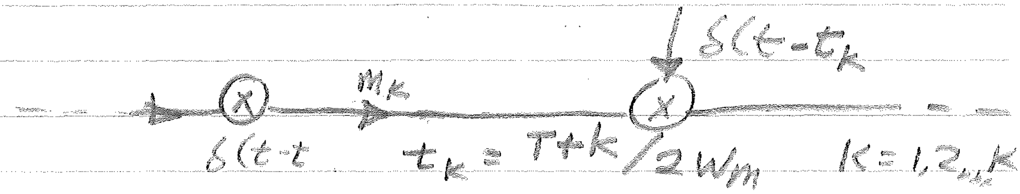
RECEIVER ESTIMATES $\hat{m}_K$, AND THEN CONSTRUCTS ESTIMATED WAVEFORM: $\hat{m}(t) = \sum_{K=1}^{K} \hat{m}_K \psi_K(t)$

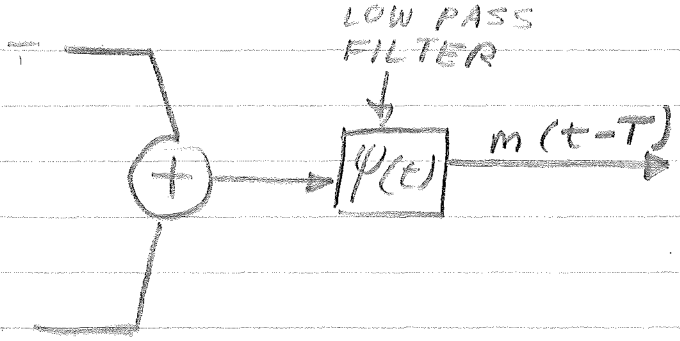
(NOTE $\hat{m}(t)$ IS BANDLIMITED)

MAXIMUM LIKLIHOOD

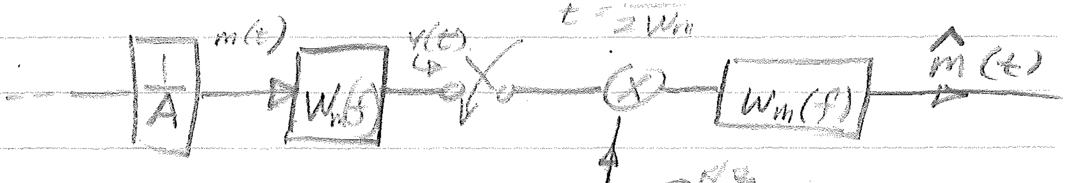








MAY ALSO REPLACE ϕ BY ψ, AND REPLACE BY LP FILTER



$$-\frac{K}{2} \le k \le \frac{K}{2} \quad \sum_{-K/2}^{K/2} \frac{1}{2W_m}\delta\left(t-\frac{k}{2W_m}\right)$$

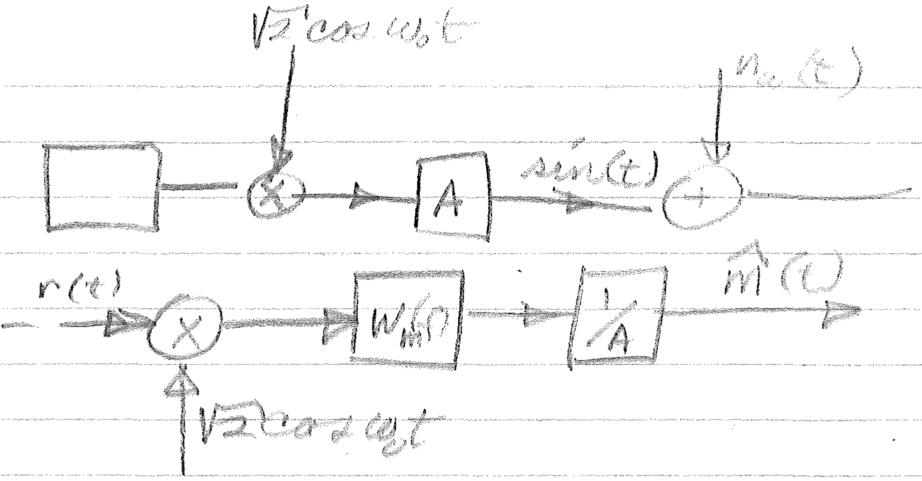
$$r(t) = \int_{-\infty}^{\infty} m(\tau)\cdot\sqrt{2W_m}\,\varphi(t-\tau)d\tau$$

$$= m_k\sqrt{2W_m} \sum_{-K/2}^{K/2} \int_{-\infty}^{\infty} \varphi(\tau - k/2W_m)\,\psi(t-\tau)d\tau$$

$$\Rightarrow r\left(\frac{k}{2W_m}\right) = m_k\sqrt{2W_m} \sum_{-K/2}^{K/2} \varphi\left(\tau-\frac{k}{2W_m}\right)\psi\left(\frac{k}{2W_m}-\tau\right)d\tau$$

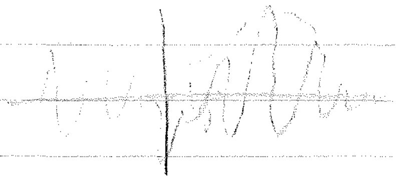
$$\psi(t) = \varphi(-t)$$

$$\Rightarrow r\left(\frac{k}{2W_m}\right) = \sqrt{2W_m}\, m_k$$



MAXIMUM LIKLIHOOD: DSB-SC.

DSB WITH CARRIER



READ 8-2 SKIP MMSE

611-619 . 621-627

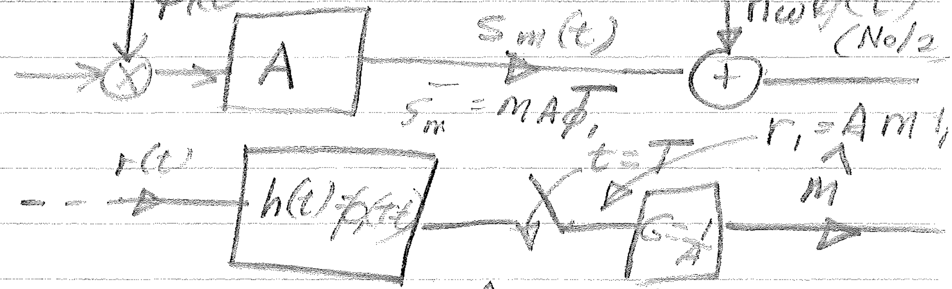
SKIP SHRESHOLD 639-642

A.L. 8.4 DUE 6/6

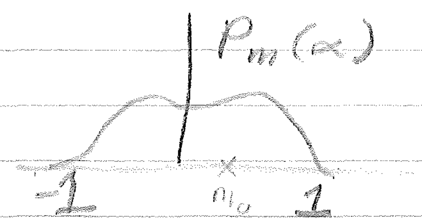
MON

TWISTED MODULATION:
(NON LINEAR)

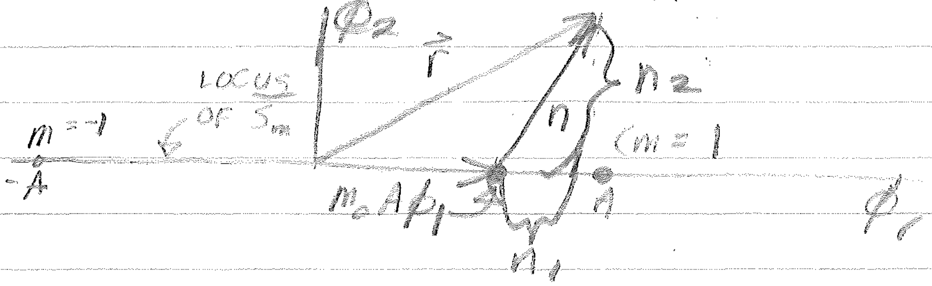
LINEAR MODULATION: $\overline{S}_m = mA\overline{\phi}_1$



$\hat{m} = m + n_1/A$



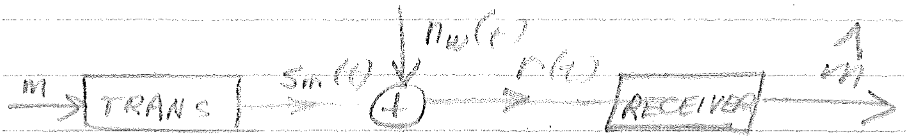
$$\overline{\varepsilon^2} = \overline{(\hat{m}-m)^2} = \left(\frac{\overline{n_1^2}}{A^2}\right) = \frac{N_0/2}{A^2}$$



$A = \left|\frac{d\overline{S}_m}{dm}\right|$ (THE "STRETCH OF THE SIGNAL LOCUS

DEFINE STRETCH:

$S \triangleq \left|\frac{d\overline{S}_m}{dm}\right|$ $\quad \varepsilon^2 = \frac{N_0/2}{S^2}$ FOR LINEAR MOD. $A = S$

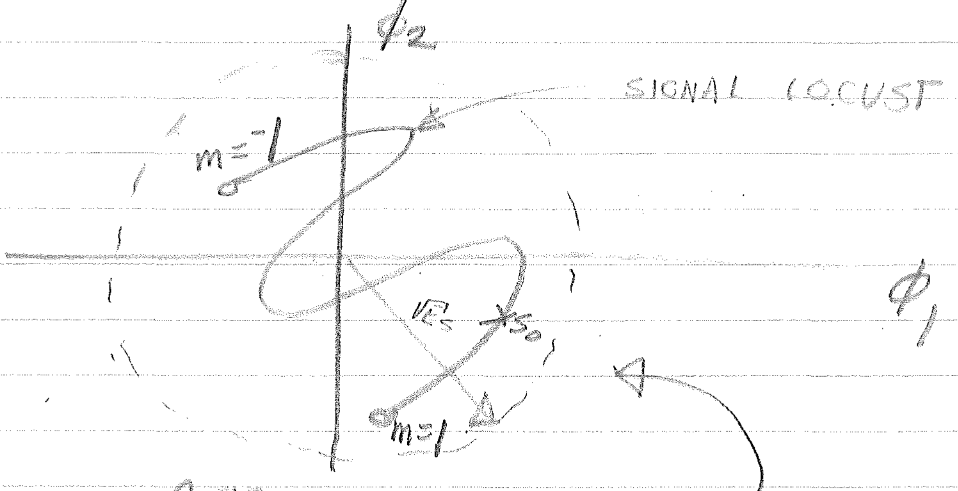


$s_m(t) = a(m)\phi_1(t) + a_2(m)\phi_2(t)$

$\delta_{ij} = \int_{-\infty}^{\infty} \phi_i(t)\phi_j(t)dt$

$\Rightarrow s_m(t) = a_1(m)\overline{\phi}_1 + a_2(m)\overline{\phi}_2$

FOR LINEAR MOD., $a_1(m)$ and $a_2(m)$ WOULD BE LINEAR FUNCTIONS OF M, FOR NON-LINEAR (TWISTED) COMM $a_1(m)$ & $a_2(m)$ ARE NON LINEAR FUNCTIONS OF M



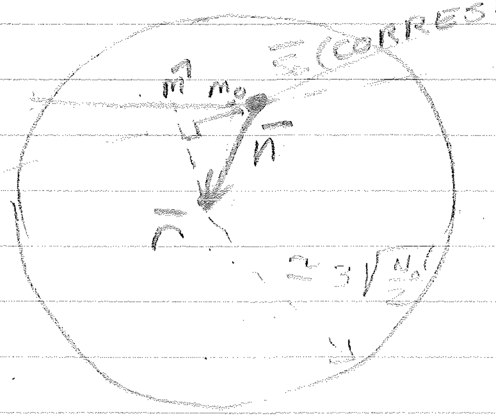
IF $\int_{-\infty}^{\infty} s^2_m(t)dt \leq E_s$

ASSUME:
$S = \left|\frac{d s_m}{dm}\right| = \frac{L}{2}$, A CONSTANT INDEPENT OF M. L = TOTAL LENGTH OF LOCUST.

$S = \frac{L}{2}$

<u>WEAK NOISE SUPPRESION</u>

$$\overline{\varepsilon^2} = \frac{N_0/2}{S^2}$$



ε (CORRESPONDS TO TRAS INPUT N_0)

$$\overline{S_m} \simeq \overline{S_0} + (m - m_0)\overline{\dot{S_0}}$$

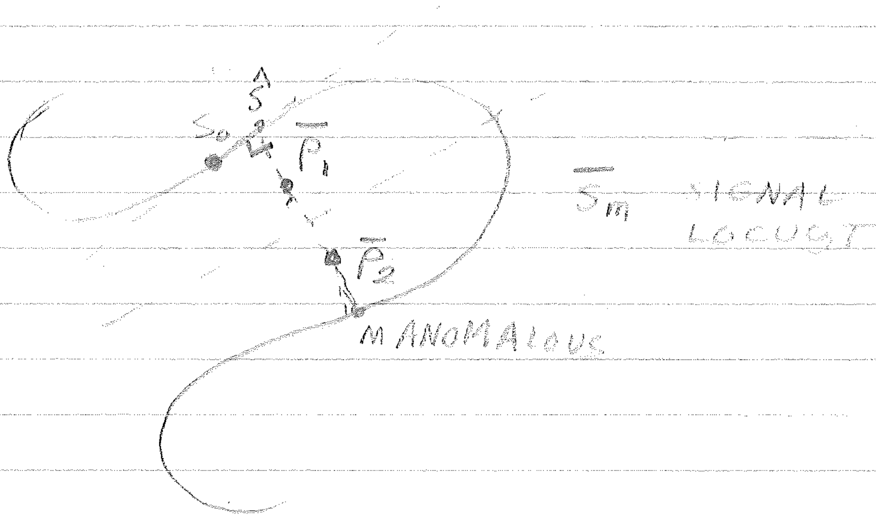
$$\Rightarrow \dot{S_0} = \frac{dS_m}{dm}\bigg|_{\substack{m = m_0 \\ \text{OR } S_m = S_0}}$$

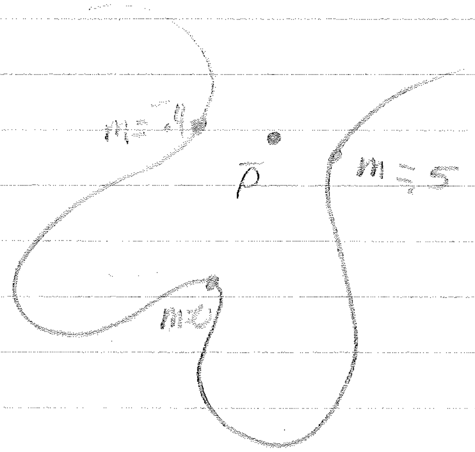
CONDITIONAL: $\overline{\varepsilon^2}$

$$E\left[(m - \hat{m})^2 / m = m_0\right] = \frac{N_0/2}{|\dot{S_0}|^2}$$

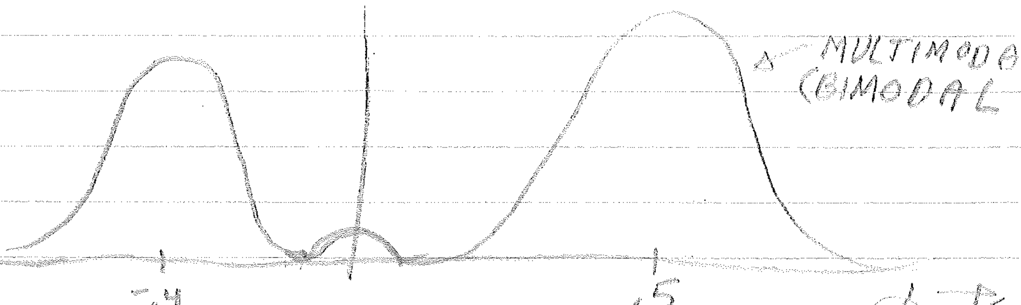
UNCONDITIONAL $\overline{\varepsilon^2}$

$$\overline{\varepsilon^2} = \frac{N_0/2}{S^2} = \frac{N_0/2}{L^2/4}$$





$P_m(\alpha|\bar{r}=\bar{\rho})$ APOSTECRIA



$$E[m|\bar{r}=\bar{\rho}]=\int_{-\infty}^{\infty}\alpha p_m(\alpha|\bar{r}=\bar{\rho})d\alpha$$

MAXIMUM LIKIHOOD RECEIVER DESIGN

FOR ADDITIVE W.G.N.

$$p_{mr}(\alpha,\rho) = p_r(\rho \mid m = m_0)\, p_m(\alpha)$$

$$= p(\rho)\, p_m(\alpha \mid r = \rho)$$

$$\Rightarrow p_m(\alpha \mid r = \rho) = \frac{p_r(\rho \mid m = \alpha)\, p_m(\alpha)}{p_r(\rho)}$$

$$p_r(\bar{\rho} \mid m = \alpha) \propto e^{-|\rho - s_\alpha|^2 / N_0}$$

MAXIMIZE $\rho \cdot s_\alpha - \frac{1}{2} E_\alpha$

WISH NO MAXIMIZE

$$(\bar{\rho} - s_\alpha)(\bar{\rho} - s_\alpha) = |\rho|^2 - |s_\alpha|^2 + 2\bar{\rho} \cdot \bar{s}$$

$$\underset{\text{NOT } f(\alpha)}{\big\downarrow} \qquad \overset{\shortparallel}{E_\alpha}$$

$$\rho \cdot s_\alpha - \frac{1}{2} E_\alpha = \int_{-\infty}^{\infty} \rho(t)\, s_\alpha(t)\, dt - \frac{1}{2} E_\alpha$$

CURRELATION DETECTION

LINEAR MODULATION;

$$s_\alpha(t) = \alpha A \phi(t)$$

THUS $\int_{-\infty}^{\infty} \rho(t)\, s_\alpha(t)\, dt - \frac{1}{2} E_\alpha$

$$= \alpha A \int_{-\infty}^{\infty} \rho(t) \phi(t) - \frac{1}{2} E_\alpha$$

$$= \alpha A \rho - \frac{1}{2} \alpha^2 A^2$$

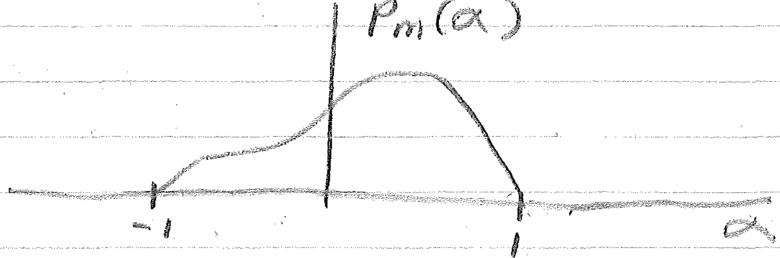
DIFFERENTIATE TO FIND MAX

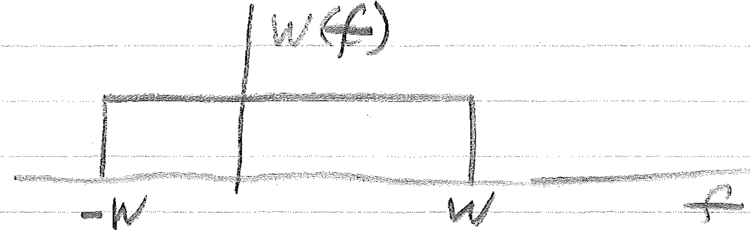
$$\Rightarrow \hat{\alpha} = P/A = \alpha' A$$

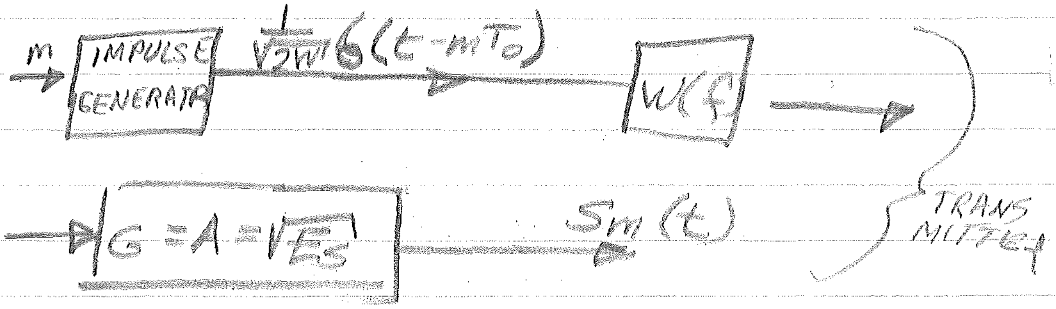
MON

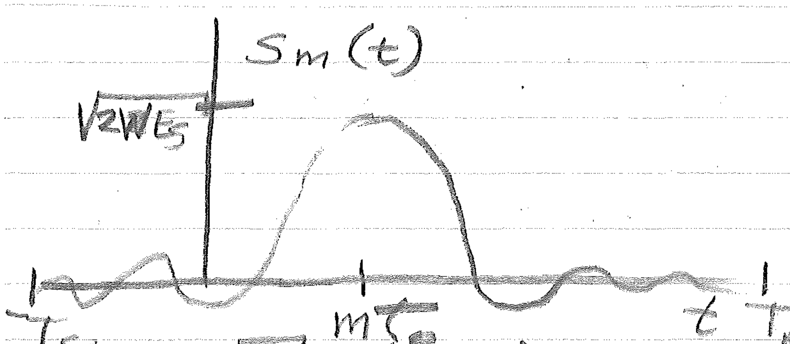
PULSE POSITION MODULATION

$$\left[\overline{\mathcal{E}^2} = \frac{N_0/2}{S^2} = \left(\frac{N_0/2}{2E_s}\right) \text{FOR LINEAR MODULATION}\right]$$









$S_m(t) = \sqrt{E_s}\,\delta(t-mT_0)$; $T_0 >> \frac{\,}{2W}$

$\phi(t) = \sqrt{2W}\,\frac{\sin 2\pi W t}{2\pi W t}$ $\quad -W < t < W$

TRANSMITTED ENERGY $|\bar{S}_m|^2 = E_S$

MAXIMUM LIKELIHOOD RECEIVER:

$p_r(\bar{\rho}|m=\alpha)$ IS TO BE MAXED

$$\alpha \int_{-\infty}^{\infty} \rho(\gamma) s_m(\gamma)\, d\gamma \qquad p_m(\alpha|r=\rho) = \frac{p_r(\rho|m=\alpha)p_m}{p_r(\rho)}$$

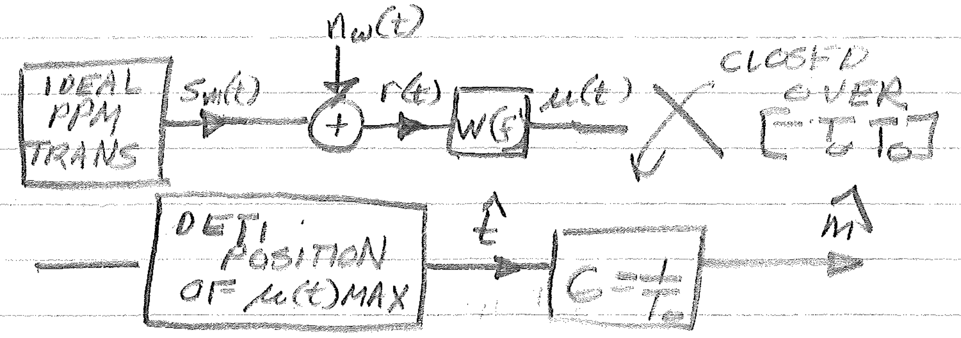
$$= \alpha \sqrt{E_s} \int_{-\infty}^{\infty} \rho(\gamma)\phi(\gamma - \alpha T_0)\, d\gamma$$

THIS IS THE OUTPUT OF A FILTER MATCHED TO $\phi(-t)$

ie $h(t) = \phi(-t)$ WHEN INPUT IS $\rho(t)$ $\xrightarrow{\rho(t)}\boxed{h(t)}\longrightarrow$

$$\text{OUTPUT} = \int_{-\infty}^{\infty} \rho(\tau) h(t-\tau)\, d\tau$$
$$= \int_{-\infty}^{\infty} \rho(\tau) \phi(\tau - t)\, d\tau$$
$$= \int_{-\infty}^{\infty} \rho(\tau) \phi(\tau - \alpha T_0)\, d\tau$$

SINCE $\phi(t) = \phi(-t)$ (FOR A LP FILTER), A MAXIMUM LIKELIHOOD RECEIVER PASSES $\rho(t)$ THRU A LP FILTER $W(f)$, THEN DETERMINES THE TIME INSTANT, $\hat{t}$, $-T_0 \le t \le T_0$ AT WHICH THE OUTPUT $\hat{t}$ IS MAX, AND SETS $\hat{M} = \hat{t}/T_0$



5.79a) $$S^2 = \int_{-\infty}^{\infty} \left[\frac{\delta}{\delta m} s_m(t)\right]^2 dt$$

$$s_m(t) = \sqrt{E_s}\,\phi\,(t - mT_0)$$

$$\frac{\delta s_m(t)}{\delta m} = -\sqrt{E_s}\,T_0\,\phi'(t - mT_0)$$

$$\Rightarrow S^2 = -\sqrt{E_s}\,T_0 \int_{-\infty}^{\infty} [\phi'(t m T_0)]^2 dt$$

$$\phi(t) = \sqrt{2W}\ \frac{\sin 2\pi W t}{2\pi W t}$$

USING PARSIVAL'S THEM:

$$S^2 = E_s T_0^2 \int_{-W}^{W} (2\pi f)^2 \frac{1}{2W}\,df$$

$$S^2 = \frac{E_s}{3}(2\pi T_0 W)^2$$

NOTE: S^2 INDEPENDENT OF m

$$\Rightarrow \overline{\varepsilon^2} = \frac{N_0/2}{S^2} = \frac{12}{\pi^2}\left(\frac{1}{4T_0W}\right)^2 \frac{N_0}{2E_s}$$

FOR PAM $= \frac{N_0}{2E_s}$

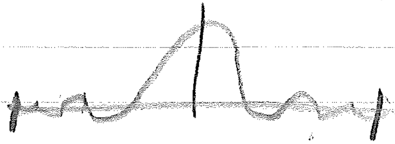
FOR LIN MOD.

$$(m_1 + m_2)\phi(t) = m_1\phi(t) + m_2\phi(t)$$

$$\varepsilon^2 = N_0/2E_s$$

CONSIDER $4T_oW$; SIGNAL BW = W

SIGNAL INTERVAL $T_S = 2T_0$



$\Rightarrow \underbrace{4T_oW}$ = TWICE BW. X SIGNAL INTERVA

SAME # ~~FROM~~ OF SAMPLES TO

COMMUN. $-T_0 < s(t) < T_0$ AND $-W < S(f) < W$

CALLED EFFECTIVE DIMENSIONALITY

D $4T_0W = \beta$

$$\therefore \overline{\varepsilon^2} = \frac{12}{\pi^2}\frac{1}{\beta^2}\left(\frac{N_0}{2E_s}\right)$$

WAVEFORM COMM. WITH P.P.M.

LOW PASS WAVEFORM

$$m(t) = \sum_{k=-\infty}^{\infty} m_k\phi_k(t);\ |f| < W_m$$

ASSUME $|m_k| \leq 1\ \forall k$

WITH P.P.M., EACH MESSAGE, m_k IS TRANSMITTED AND ESTIMATED BY A MAXIMUM LIKELIHOOD RECEIVER IN SUCCESSION, AND

PRODUCES $\hat{m}_k$. THE RECEIVER CONSTRUCTS

$$\hat{m}(t) = \sum_{k=-\infty}^{\infty} \hat{m}_k \psi(t)$$

$$\hat{m}_k = m_k + n_k$$

$$\Rightarrow \hat{m}(t) = m(t) + n(t)$$

$$n(t) = \sum_{k=-\infty}^{\infty} n_k \psi_k(t)$$

= STATIONARY GAUSSIAN PROCESS WITH SPECTRAL DENSITY

$$S_n(f) = \begin{cases} \overline{\epsilon^2} & |f| \leq W_m \\ = 0 & \text{ELSEWHERE} \end{cases}$$

AVE. OUTPUT NOISE PWR

$$= \overline{n^2(t)} = 2W_m \underline{S_n(f)}$$

$$= 2W_m \overline{\epsilon^2}$$

OR $\overline{n^2(t)} = \frac{1^2}{\pi^2} \frac{1}{\beta^2} \frac{N_o W_m}{E_s}$

FOR LINEAR MOD:

$$\overline{n^2(t)} = N_o W_m / E_s$$

TIME AVAILABLE FOR TRANSMISSION OF EACH m_k IS THE SAMPLING INTERLAL $\frac{1}{2W_m}$. THUS, MAX ALLOWABLE POSITION DEVIATION FOR EACH TRANSMISSION IS $2T_0 \leq \frac{1}{2W_m}$. IF WE SET $2T_0 = 1/2W_m$, THEN WE GET

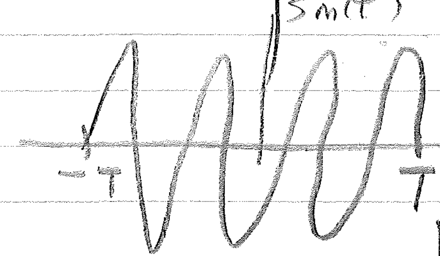
$$4T_0 W = B = W/W_m = \overset{B.W}{\text{EXPANSION RATIO}}$$

FREQUENCY POSITION MODULATION

$$S_m(t) = \sqrt{E_s}\,\phi_m(t)$$

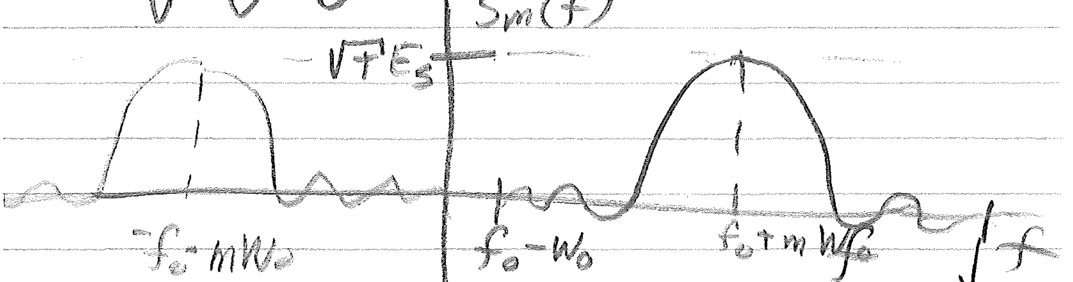
$$\phi_m(t) = \frac{1}{\sqrt{T}} \cos 2\pi(f_0 + W_0 t);\ -T < t < T$$

$$= 0 \quad \text{OTHERWISE}$$



$f_0 >> W_0$

$-1 \leq m \leq M$



$\hat{f} = f_0 + \hat{m} W_0$

$|\leftarrow \frac{T}{2} \rightarrow|$

$f_0 + W_m$

$B = 4TW_0$

$$\hat{m} = \frac{\hat{f} - f_0}{W_0}$$

$$S^2 = \frac{E_s}{3}(2\pi T W_0)^2$$

$$\overline{n^2(t)} = \frac{12}{\pi^2}\left(\frac{W_m}{W_0}\right)^2 \frac{N_0 W_m}{E_s}$$

MON

FM

$$S_m(t) = A\sqrt{2}\cos 2\pi\left[\underbrace{f_0 t + W_1 \int m(t)\,dt}_{\text{INSTANTANEOUS PHASE}}\right]$$

$$f_{INST} = \frac{d}{dt}\left[f_0 t + W \int m(t)\,dt\right]$$
$$= f_0 + W_1\, m(t)$$

SPECTRUM: LET $m(t) = \cos 2\pi W_m t)$

$$w_0 = 2\pi f_0 \quad w_m = 2\pi W_m$$

$$S_m(t) = A\sqrt{2}\cos\left(w_0 t + \frac{W_1}{W_m}\sin w_m t\right)$$

$$\beta = W_1 / W_M$$

$$S_m(t) = A\sqrt{2}\left[\cos(\beta \sin wt)\cos w_0 t - \sin(\beta \sin w_m t)\sin w_0 t\right]$$

$$J_K(B) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i}$$

3-19-71

MON 2.12, 2.23, 2.24 COMM PROB " EX ON 78

(3.4 3.5) 3.4 3.6 3.7 3.8

ALL OF CHAPT. 4 CORR. REC. MATCHED FILTR

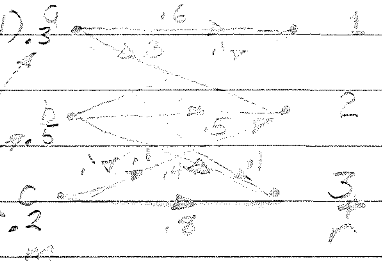
SKIP CHAPT 5, 6

~~CHAPT 7: 1, 2~~

CHAPT 8: 1, 2, 3, 5

LECTURE

2.11)



APRIORI PROB = PROB. THAT a, b, c WERE SENT AFTER HEARING A 1, 2, 3

$P[m_k]$ = APRIORI PROB. OF k^{TH} MESSAGE

$P[r_j/m_k]$ = CONDITIONAL PROB OF RECEIVING THE j^{TH} r GIVEN OR CONDITIONED UPON TRANSMITTING THE k^{TH} MESSAGE

$$P[m_k]P[r_j|m_k] = P[m_k,r_j] = P[m_k/r_j]P[r_j]$$

$P[m_k, r_j]$ = PROB. OF THE JOINT EVENT m_k TRANSMITTED AND r_j IS RECEIVED (APOSTERORI PROB) (CONDITIONAL)

BAYES RULE

$$P[m_k|r_j] = \frac{P[m_k r_j]}{P[r_j]}$$

$$P[m_k | r_j] = \frac{P[m_k, r_j]}{P[r_j]}$$

$\uparrow$ FIXED $\qquad \uparrow$ FIXED

WANT VALUE OF m_k TO MAX. $P[m_k | r_j]$

$$\frac{P[m_k, r_j]}{P[r_j]} = \frac{P[m_k]\,P[r_j / m_k]}{P[r_j]}$$

MAXIMIZING $\Longrightarrow$ MAXIMIZING

$\underline{P[m_k]\,P[r_j / m_k]}$

$P[r_j] = P[r_j, m_1] + P[r_j, m_2] + P[r_j, m_3]$

$P[1] = P[1, a] + P[1, b] + P[1, c]$

$= (.3)(.6) + (.5)(.1) + (.2)(.1)$

$P[2] = P[2, a] + P[2, b] + P[2, c]$

$= (.3)(.3) + (.5)(.5) + (.2)(.1)$

$= .36$

$P[3] = .39$

$P[1,a] = .18\,;\quad P[1,b] = .05 \quad P[1,c] = .02$

$P[2,a] = .09 \quad P[2,b] = .25 \quad P[2.c] = .02$

$P[3,a] = .03 \quad P[3,b] = .20 \quad P[3,c] = .16$

RECEIVER DECISION RULE

$\hat{m}(1) = a \qquad (.18)$

$\hat{m}(2) = b \qquad (.25)$

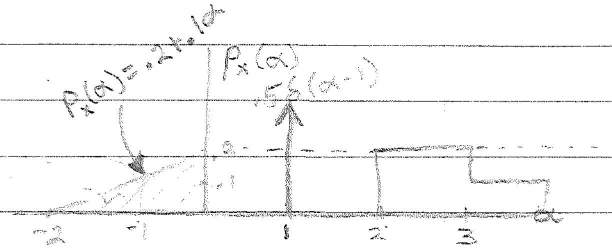
$\hat{m}(3) = b \qquad (.20)$

b) $P[C] = .18 + .25 + .20 = .63 \Rightarrow P[E] = .37$

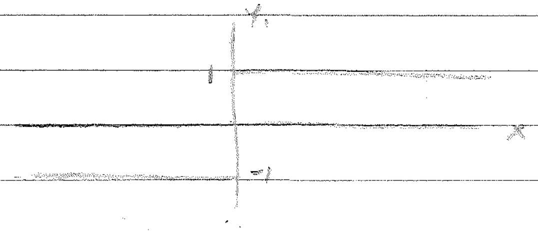
c) b DOMINATES $\Longrightarrow P[E] = .50$

$\hat{m}(\cancel{1,2,}3) = b$

2.18)

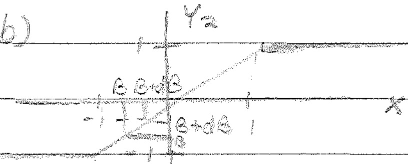


a)



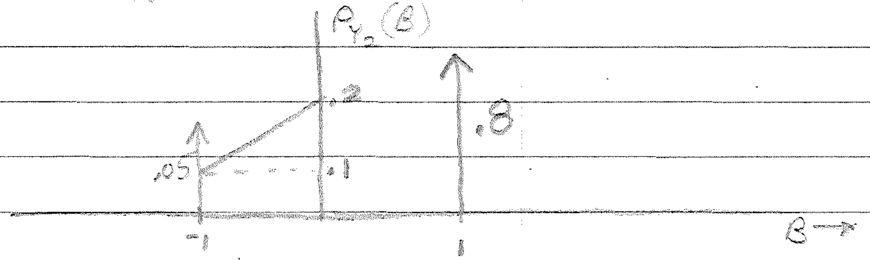
$$P_{Y_1}(\beta)=.2\delta(\beta+1)+.8\delta(\beta-1)$$

b)

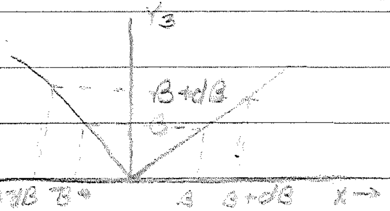


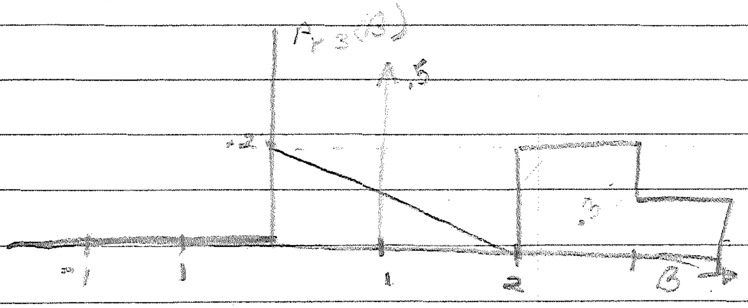
FOR $-1<\beta<0$, $P_{Y_2}(\beta)=.2+.1\beta$

$$P_{Y_2}(\beta)=.05\delta(\beta+1)+.8\delta(\beta-1)+\overbrace{.2+.1\beta}^{-1<\beta<0}$$

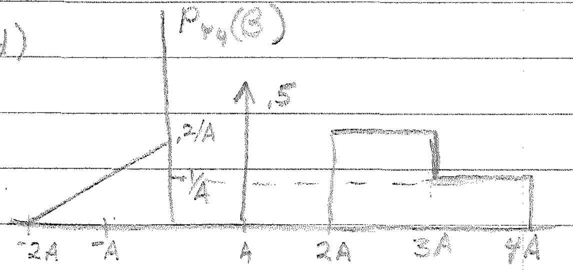


c)





d)



e) $P_{Y_5}(B) = .5\delta(B-1) + .1\delta(B-2) + .15\delta(B-3)$
$+ .05\delta(B-4) + .0875(B-4)$
$+ .1\delta(B+1) + .0125\delta(B+2)$

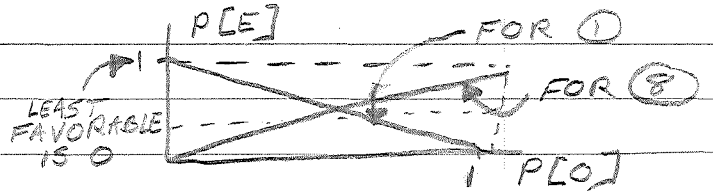
3-23-71

2.12) $(a,b,c) = \overset{①}{(0,0,0)}, \overset{②}{(001)}, \cdots, \overset{⑧}{(111)}$

& PROB. OF MAKING CORR. CHOICE

HINT: (0,0,0) $P[C] = P[0$ WAS TRANSM$]$

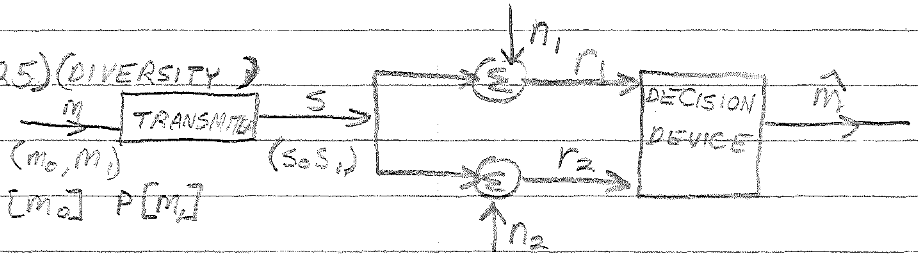
$P[\varepsilon_1] = 1 - P[0]$



$P[\varepsilon_8] = P[0]$

⑤

2.25) (DIVERSITY)



$s_0 = (-1)^0 \sqrt{E} = \sqrt{E} \qquad s_1 = (-1)^1 \sqrt{E} = -\sqrt{E}$

$\ni E = \text{ENERGY} \propto V^2$

$r_1 = s + n_1 \; ; \; r_2 = s + n_2$

BAYES MIXED RULE:

$p_r(\rho/m)\, P[m] = P[m/r=\rho]\, p_r(\rho)$

$= p_r(\rho, m)$ (JOINT DISTRIBUTION FUNCTION OF RANDOM EVENT r & EVENT m)

WANT TO MAXIMIZE A'POSTERORI PROB:

$$P[m_i/r=\rho] = \frac{p_r(\rho/m_i)\, P[m_i]}{p_r(\rho)}$$

$P[m_0/r=\rho] > P[m_i/r=\rho]$

$\Rightarrow p_r(\rho/m_0)\, P[m_0] > p_r(\rho/m_i)\, P[m_i]$

$r_i = s_i + n \Rightarrow p_r(\rho/m_i) = ?$

$P[\rho < r < \rho + d\rho \,/\, m = m_i]$

$= P[\rho < s_i + n < \rho + d\rho \,/\, m_i]$

$= P[\rho - s_i < n < \rho - s_i + d\rho \,/\, m = m_i]$

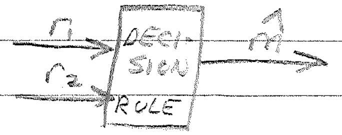
$= p_n(\rho - s_i / m = m_i)\, d\rho$

m_i IND OF n $\Rightarrow$

$= p_n(\rho - s_i)\, d\rho$

$$\boxed{\therefore p_r(\rho / m = m_i) = p_n(\rho - s_i)}$$

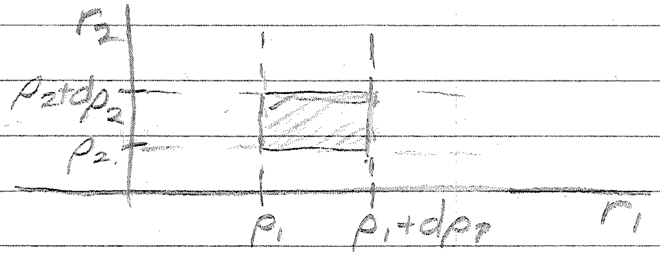
WE MUST EXAMINE THE TWO VARIABLE CASE



WANT $P[m_0 | r_1 = \rho_1, r_2 = \rho_2]$

$P[m_0 | r_1 = \rho_1, r_2 = \rho_2] > P[m_1 | r_1 = \rho_1, r_2 = \rho_2]$

$\Rightarrow p_{r_1,r_2}(\rho_1 \rho_2 | m_0) \overset{P[m_0]}{>} p_{r_1 r_2}(\rho_1 \rho_2 | m_1) P[m_1]$



BY ANALOGY

$p_{r_1 r_2}(\rho_1 \rho_2 | m_i) = p_{n_1 n_2}(\rho_1 - s_i, \rho_2 - s_2 | m_i)$

(n_1 & n_2 IND FROM m_i

$= p_{n_1}(\rho_1 - s_i) p_{n_2}(\rho_2 - s_i)$

mean $= s^i$

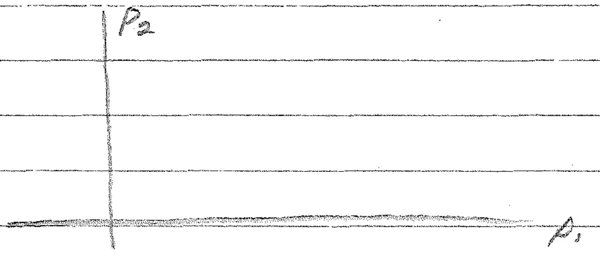
$p_{n_i}(\rho_1 - s_i) = \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-(\rho_i - s_i)^2 / 2\sigma_i^2}$

$p_{n_2}(\rho_2 - s_i) = \frac{1}{\sqrt{2\pi}\,\sigma_2} e^{-(\rho_2 - s_i)^2 / 2\sigma_2^2}$

$p_{r_1 r_2}(\rho_1 \rho_2 | m_0) P[m_0] = \frac{P[m_0]}{2\pi \sigma_1 \sigma_2} e^{-\left(\frac{(\rho_1 - s_0)^2}{2\sigma_1^2} + \frac{(\rho_2 + s_0)^2}{2\sigma_2^2}\right)}$

$p_{r_1 r_2}(\rho_1 \rho_2 | m_1] P[m_1] = \frac{P[m_1]}{2\pi \sigma_1 \sigma_2} e^{-\left[\frac{(\rho_1 - s_1)^2}{2\sigma_1^2} + \frac{(\rho_2 - s_0)^2}{2\sigma_2^2}\right]}$

⑦



Let $\rho_1 = \rho_2$

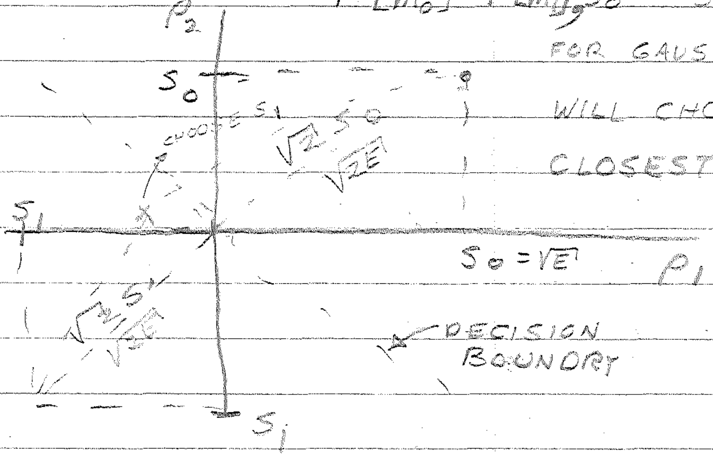
$$p_{r_1 r_2}(\rho_1, \rho_2 / m_0)\, P(m_0] = \frac{P(m_0)}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}\left[(\rho_1 - s_0)^2 + (\rho_2 - s_0)^2\right]}$$

$$P_{r_1 r_2}(\rho_1, \rho_2 / m_1)\, P[m_1] = \frac{P[m_1]}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}\left[(\rho_1 - s_1)^2 + (\rho_2 - s_1)^2\right]}$$

ALL THAT IS NEEDED IS EXPONENENS

$$(\rho_1 - s_0)^2 + (\rho_2 - s_0)^2 > (\rho_1 - s_1)^2 + (\rho_2 - s_1)^2$$

$P[m_0] = P[m_1]; s_0 = -s_1$



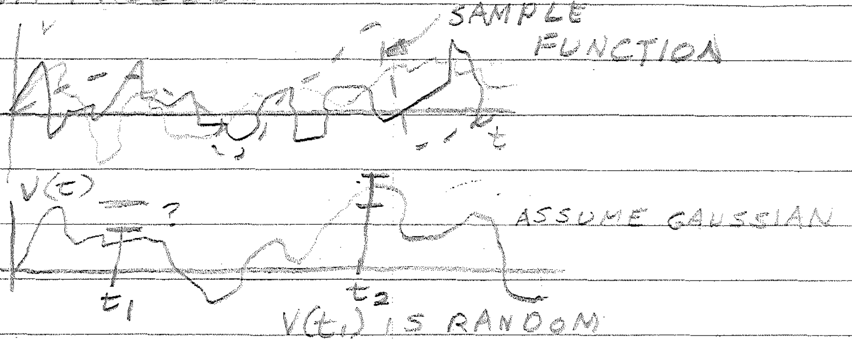
FOR GAUSSIAN NOISE, WILL CHOOSE SIGNAL CLOSEST TO VOLTAGE

THE FARTHER APART THE POINTS, THE BETTER THE DECISION

3-27-71

CHAPT. 3

RANDOM PROCESS:



$$\rho_{x(t)}(\underset{\uparrow}{\alpha}, t_1)$$

SINGLE VAR. = V at t_1

$$\rho_{x(t_1),x(t_2)}(\alpha_{t_1}, \beta_{t_2})$$

$$\vdots$$

$$\downarrow$$

$$\rho_{x(t_1)\ldots x(t_n)}(\overbrace{\alpha_1, \ldots, \alpha_n}^{\beta})$$

IF EACH IS GAUSSIAN $^{\alpha}$ IS GAUSSIAN

I~~F~~ P IS DEPENDENT ONLY ON Δt,

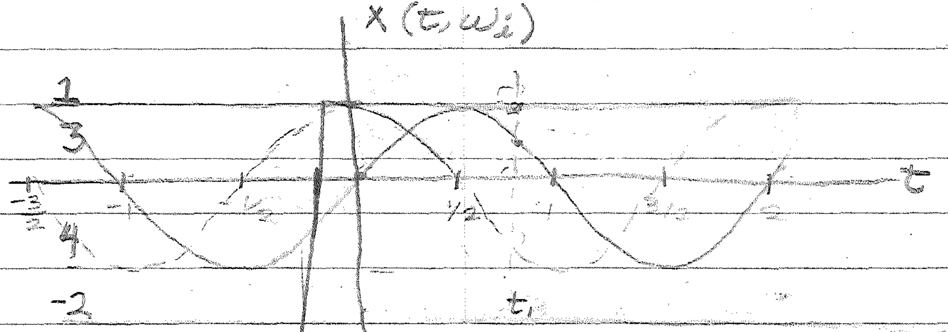
SYSTEM IS STATIONARY

NEED ONLY $\overline{x^2}$ & $\overline{X}$ TO DEFINE GAUSSIAN PRODUCT

READ: 3.1, $\overset{\sim}{3.2}$, 3.4, 3.5

(3.1) $x(\omega_1, t) = 1$ $x(\omega_3 t) = \sin \pi t$

$x(\omega_2 t) = {}^-2$ $x(\omega_4, t) = \cos \pi t$



a) $P[\omega_i] = 1/4 \ \forall i$

$$P_{x(t_1)}(\alpha) = \frac{1}{4}\delta(\alpha-1) + \frac{1}{4}(\delta+2) + \frac{1}{4}\delta(\alpha - \sin\pi t_1) + \frac{1}{4}\delta(\alpha - \cos\pi t_1)$$

$P_{x(t_1)}(\alpha)$ CHANGES WITH TIME ⇒ NOT STATIONARY

b) $\bar{x}(t) = \int_{-\infty}^{\infty} \alpha \, P_{x(t)}(\alpha) \, d\alpha$

$$= \frac{1}{4}(1 - 2 + \sin\pi t + \cos\pi t)$$

$$\overline{x(t_1) x(t_2)} = \int_{-\infty}^{\infty} \int \alpha \beta \, P_{x(t_1), x(t_2)} \, d\alpha \, d\beta$$

$= R(t_1, t_2) =$ CORRELATION FUNC.

TO CALCULATE: $P_{x(t_1) x(t_2)}(\alpha, \beta)$

$$= P_{\bar{x}(t)}(\bar{\alpha}) = P_{x(t_1)}(\alpha) P_{x(t_2)}[\beta / x(t_1 = \alpha]$$

WHAT IS $P_{x(t_2)}(\beta / x(t_1) = \alpha)$?

LET $t_1 = \frac{1}{4}$, THEN ONLY VALUES OF α ARE $^-1, ^-2, \pm .707$

THEN, IF $\alpha = 1$, $P_{x(t_2)}(\beta / x(t_1) = 1)$

$= \delta(\beta - 1)$ (MUST STAY ON SAME FUNCTION)

THEN, IF $\alpha = {}^-2$ $P_{x(t_2)}(\beta / x(t_1 = {}^-2)$

$= \delta(\beta + 2)$

THEN, IF $\alpha = .707$, $P_{x(t_2)}(\beta / x(t_1) = .707)$

$= \delta(\beta - \sin \pi t_1)$

THEN, IF $\alpha = .707$, $p_{x(t_2)}\left(\beta / x(t_1) = .707\right)$

$= \delta(\beta - \cos \pi t_2)$

$\Rightarrow \overline{x(t_1)(x(t_2)} = \overline{x(-\frac{1}{4})\, x(-\frac{1}{4})}$

$= \int\int_{-\infty}^{\infty} \alpha \cdot \beta \cdot \frac{1}{4}\left[\delta(\alpha-1) + \delta(\alpha+2)\right.$

$\left. + \delta(\alpha+.707) + \delta(\alpha-.707\right]$

$p\left[\beta / x(-\frac{1}{4}) = \alpha\right)\, d\alpha\, d\beta$

$= \int_{-\infty}^{\infty}\left[\beta \cdot \frac{1}{4}\left[p_{x(t_2)}\left(\beta / x(-\frac{1}{4}) = 1\right)\right.\right.$

$- 2 p_{x(t_2)}\left(\beta / x(-\frac{1}{4}) = -2\right)$

$-.707\, p_{x(t_2)}\left(\beta / x(-\frac{1}{4}) = -.707\right)$

$\left.\left. + .707\, p_{x(t_2)}\left(\beta / x(-\frac{1}{4}) = .707\right)\right]\, d\beta\right]$

$= \int_{-\infty}^{\infty} \beta\, \frac{1}{4}\left[\delta(\beta-1) - 2\delta(\beta+2)\right.$

$-.707\,\delta(\beta - \sin \pi t_2 + .707\,\delta(\beta - \cos \pi t_2) d\beta$

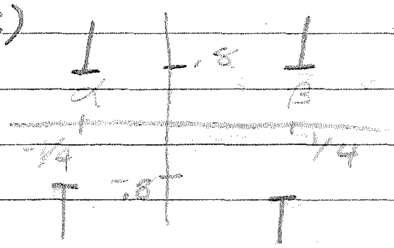
$\therefore \overline{x(t_1)x(t_2)} = \frac{1}{4} + 1 - \frac{.707}{4} \sin \pi t_2$

$+ \frac{.707}{4} \cos \pi t_2 = \frac{5}{4} + \frac{.707}{4}(\cos \pi t_2 - \sin \pi t_2)$

FOR $t_1 = -1/4$

CAN'T PUT AS $f(t_2 - t_1) \Rightarrow$ NOT SECOND ORDER STATIONARY

c)



$\text{PROB} = \int_{-.8}^{.8}\int_{-.8}^{.8} p_{x(-\frac{1}{4})}(\alpha)\, p_{x(\frac{1}{4})}\left(\beta / x(-\frac{1}{4}) = \alpha\right) d\alpha\, d\beta$

$= \int\int_{-.8}^{.8} \frac{1}{4}\left[\delta(\alpha-1)\delta(\beta-1)\right.$

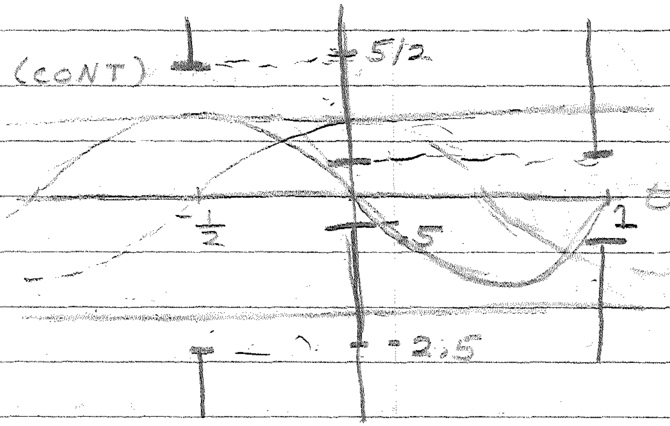
$+ \delta(\alpha+2)\,\delta(\beta+2)$

$\left. + \delta(\alpha+.707)\,\delta(\beta-.707) + \delta(\alpha-.707)\,\delta(\beta-.707)\right] d\alpha\, d\beta$

$\therefore P = \frac{1}{2}$

(CONT.)

c) (CONT)

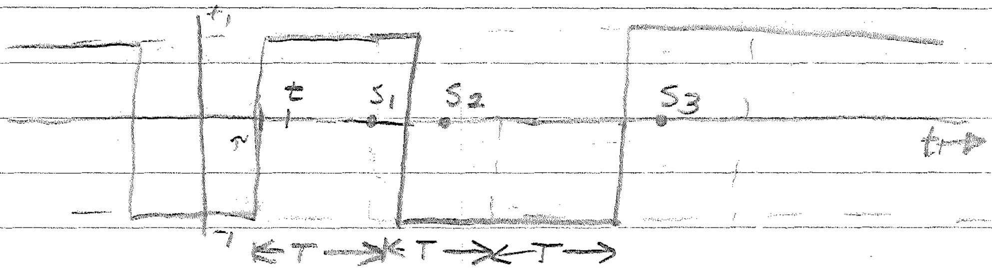


$\rho = 2/3 = \frac{8}{12}$

5-2-71 DID 3-9, 3-10

3-12)

$$x(t) = \sum_{i=-\infty}^{\infty} w_i\, \mu(t - iT - \overset{R.V.}{\tau})$$



w_i, THE VALUE OF $x(t)$ IN THE i^{TH} INTERVAL, IS IND. OF w_j, THE VALUE OF $x(t)$ IN j^{TH} INTERVAL

∴ WHEN $|t - s_3| > T$, THE CORRELATION FUNCTION

$$R_x(t, s_3) = \overline{x(t)x(S_3)} = \frac{1}{4}\underbrace{(1-1-1+1)}_{\text{PRODUCT OF ALL POSSIBLE}} = 0$$

PAIRS OF VALUES OF $X(t)$ AND $X(S_3)$

WHEN t AND S ARE IN SAME INTERVAL

EX) LET $S = S_1$

$$\Rightarrow x(t) = x(S_1) = \pm 1$$

$$\therefore \overline{x(t)x(S_1)} = R_x(t,S) = 1$$

$$\therefore \overline{x(t)x(S)} = 1 = P[t \,\&\, S \text{ ARE IN SAME INTERVAL}]$$

$$+ 0 \cdot P[t \,\&\, S \text{ ARE NOT IN SAME INTERVAL}]$$

$$= P[t \,\&\, S \text{ ARE IN SAME INTERVAL}]$$

FROM THE FIGURE, t & S WILL LIE IN THE SAME T SEC. INTERVAL IFF

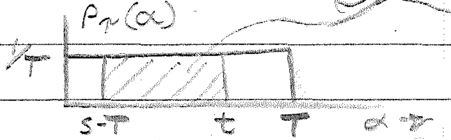
$S - T < \tau < t$. HOLDS FOR

$S > t$ (FOR EX., IN THE FIGURE)

$$P[S-T<\tau<t] = (t-S+T)\frac{1}{T} = 1 - \frac{S-t}{T}$$

$$= 1 - \frac{U}{T} \quad (U = S-t)$$

FOR $S > t$



LET $S < T$, THEN S & t BELONG TO THE SAME INTERVAL IFF

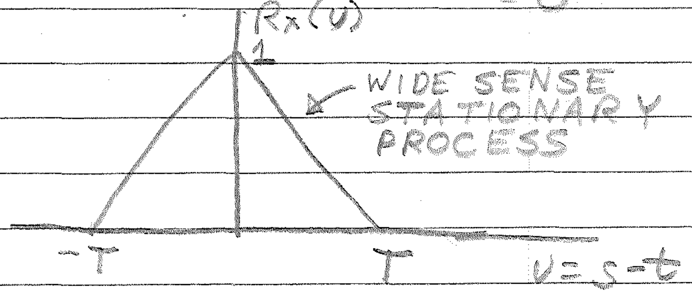
$t - T < \tau < S$

$$\Rightarrow P[t-T<\tau<S] = 1 + \frac{U}{T} \quad (U = S-T)$$

$$\therefore R_x(t,S) = R_x(S-t) = 1 + \frac{U}{T} \quad U < 0$$

$$= 1 - \frac{U}{T} \quad U > 0$$

$$= 0 \quad |U| > T$$



⑬

AMBIGUITY FUNCTIONS

$x(t)$

$x_\tau(t) = x(t+\tau)$

DEFINE A SIGNAL SPACE

$\{x(t), y(t), z(t), \xi(t) \ldots\}$

PLUS A 'METRIC OR 'DISTANCE' BETWEEN SIGNALS

$$d^{\text{ISTANCE}}(x, y) = \int_{-\infty}^{\infty} |x(t) - y(t)|^2 dt$$

$$d(x, x_\tau) = \int_{-\infty}^{\infty} |x(t) - x_\tau(t|^2 dt$$

$$= \int_{-\infty}^{\infty} [x(t) - x(t+\tau)]$$

$$[x^*(t) - x^*(t+\tau] dt$$

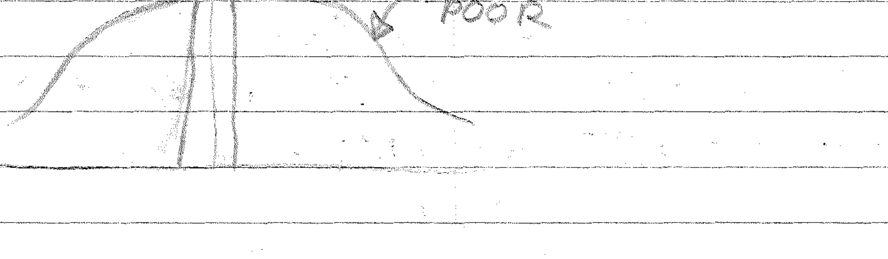
$$= \int_{-\infty}^{\infty} [\overbrace{x(t)x^*(t)}^{\text{POWER}} + x(t+\tau)x^*(t+\tau)] dt$$

$$- \int_{-\infty}^{\infty} [x(t)x^*(t+\tau) + x(t+\tau)x^*(t)] dt$$

$$\left(R_x(u) = \int_{-\infty}^{\infty} x(t)x(t+u) dt\right)$$

$$\Rightarrow d(x, x_\tau) = R_x(0) + R_x(0) - R_x(\tau) - R_x(\tau)$$

$$= 2(R_x(0) - R_x(\tau))$$



5-10-71

4-1)

MUST MAXIMIZE: $P[m_i]\,p_{\hat{r}}(\rho/\bar{s}=\bar{s})$; $i=0\,1$

FOR $i=k$, $\Longrightarrow \hat{m}=m_k$

NOW, $P[m_i]=1/2$ FOR $i=0,1$

ie $P[m_i]$ IS INDEPENDENT OF i.

HENCE, CHOOSE $\hat{M}=m_k$ IF

$p_{\bar{r}}(\bar{\rho}/s-s_i)$ IS MAX FOR $i=k$.

$$\bar{r}=n\begin{bmatrix}1\\1\end{bmatrix}+S \text{ OR } r_{1i}=n+s_{1i} \quad r_{2i}=n+s_{2i}$$

FORGET VECTOR NATURE: $r=n+s$

GIVEN p_n, WHAT IS $p_r(\rho/s=s_i)$

$n=r-s$

$$p_r(\rho/s=s_i)=p_n(\rho-s_i/s=s_i)$$

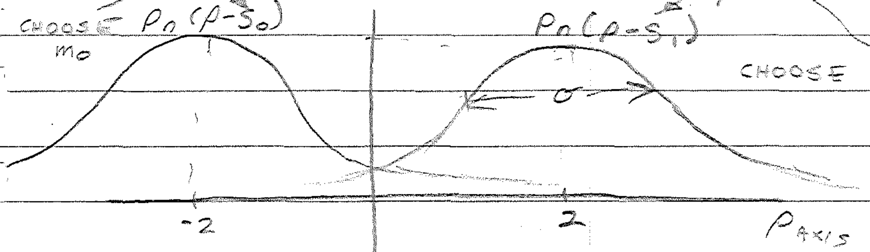
$$=p_n(\rho-s_i) \text{ (n IS STATISTICALLY IND. OF s)}$$

BECAUSE NOISE IS 0-MEAN GAUSSIAN

$$\Rightarrow p_n(\rho-s_i)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left(\frac{-1}{2\sigma^2}(\rho-s_i)^2\right)$$

$$\Rightarrow p_n(\rho-s_0)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left(\frac{-1}{2\sigma^2}(\rho+2)^2\right)$$

$$p_n(\rho-s_1)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left(\frac{-1}{2\sigma^2}(\rho-2)^2\right)$$



$$P[C/m_0]=\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{0}\exp\left(\frac{-1}{2\sigma^2}(\rho+2)^2\right)d\rho$$

$$=\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{2}\exp\left(\frac{-\sigma^2}{2\sigma^2}\right)d\sigma$$

(CONT)

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$$P[C/m_0] = \frac{1}{2} + \frac{1}{\sqrt{2\pi}\,\sigma} \int_0^2 \exp \frac{-\rho^2}{2\sigma^2} d\rho$$

$$P[C/m_1] = \frac{1}{2} + \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-2}^0 \exp \frac{-\rho^2}{2\sigma^2} d\rho$$

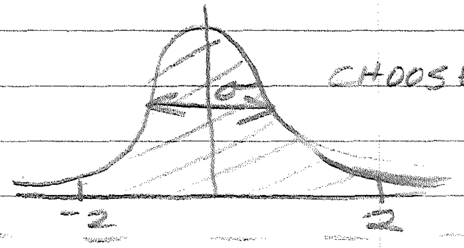
$$\Rightarrow P[C] = P[C/m_0] + P[C/m_1]\,P[m_i]$$

$$= \frac{1}{2} + \frac{1}{2}\frac{1}{\sqrt{2\pi}\,\sigma} \int_{-2}^{2} \exp\left(\frac{-\rho^2}{2\sigma^2}\right) d\rho$$

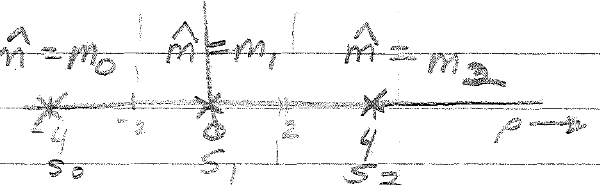
$$\overline{P[\varepsilon] = .01} \Rightarrow P[C] = .99$$

$$\Rightarrow .99 = \frac{1}{2} + \frac{1}{2}\frac{1}{\sqrt{2\pi}\,\sigma} \int_{-2}^{2} \exp \frac{-\rho^2}{2\sigma^2} d\rho$$

$$\text{or } .98 \frac{1}{\sqrt{2\pi}\sqrt{\sigma^2}} \int_{-2}^{2} \exp\left(\frac{-\rho^2}{2\sigma^2}\right. d\rho$$



CHOOSE $\sigma \Rightarrow \int_{-2}^{2} \rho = 98\%$ OF WHOLE THING



FOR OPTIMUM RECIEVER WITH

$P[m_i] = \frac{1}{3}\ \forall\ i$

$$P[\varepsilon] = 1 - \frac{1}{3}\left[P(C/m_0) + P[C/m_1] + P[C/m_2]\right]$$

INVESTIGATE

$$p_n(\rho - s_0) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(\frac{-1}{2\sigma^2}(\rho+4)^2\right) = p_r(\rho/s = s_0)$$

$$p_n(\rho - s_1) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(\frac{-1}{2\sigma^2}\rho^2\right) = p_r(\rho/s = s_1)$$

$$p_n(\rho - s_2) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(\frac{-1}{2\sigma^2}(\rho-4)^2\right) = p_r(\rho/s = s_2)$$

b) $p_n(\rho\, s_i) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp \frac{-(\rho - 1 - s_i)^2}{2\sigma^2}$

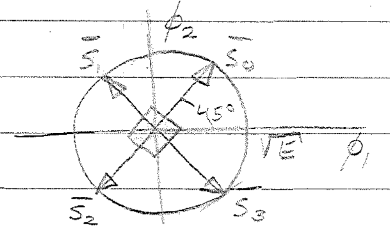
$\Rightarrow$ SHIFTED TO RIGHT ONE UNIT

(NO CHANGE IN $P[\varepsilon]$)

READ APPENDIX 4-A

⑯

4-2)



(ϕ ∓ ϕ_2 AREORTHOGONOL $\int_{-\infty}^{\infty} \phi_1 \phi_2 = 0$

$\int_{-\infty}^{\infty} \phi_1^2(t)dt = \int_{-\infty}^{\infty} \phi_2(t)dt =$ ENERGY

IF $E = 1$, ϕ_1 ∓ ϕ_2 ARE ORTHONORMAL)

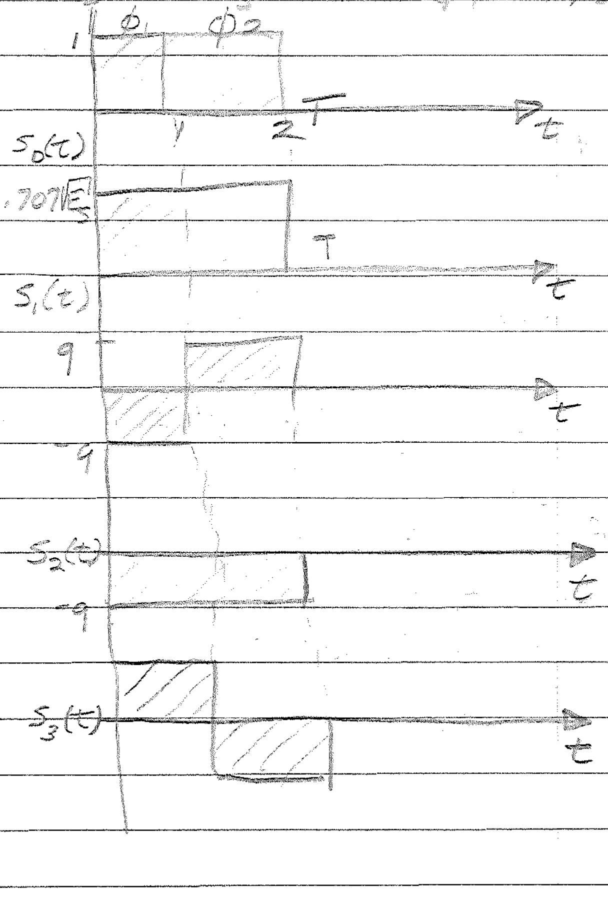
ENERGY $= E_S$

$\overline{S_0} = \sqrt{E_S}\,(.707\phi_1 + .707\overline{\phi_2})$

$\overline{S_1} = \sqrt{E_S}\,(-.707\phi_1 + .707\overline{\phi_2})$

$\overline{S_2} = \sqrt{E_S}\,(-.707\phi_1 - .707\overline{\phi_2})$

$S_3 = \sqrt{E_S}\,(.707\phi_1 - .707\overline{\phi_2})$



$q = .707\sqrt{E_S}$

$\int S_i^2(t) = E$

S_0 ORTH. TO S_1

S_0 〃 〃 S_3

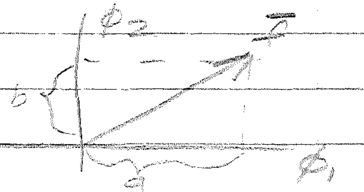
$S_1 = -S_3$

$S_0 = -S_2$

$$f(t) \Rightarrow \bar{f}$$

$\bar{\phi}_1 = \phi_1(t)$, $\phi_2(t) = \bar{\phi}_2$ ORTHONORMAL

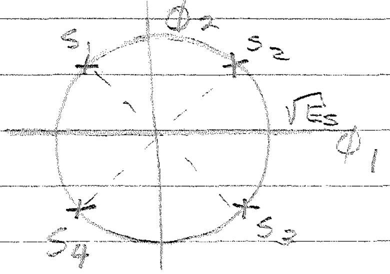
$$\bar{f} = a\bar{\phi}_1 + b\bar{\phi}_2$$

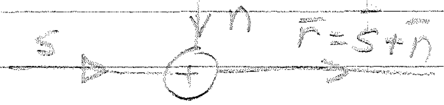


$$a = \bar{\phi}_1 \cdot \bar{f} = \int_{-\infty}^{\infty} \phi_1(t)\, f(t)\, dt$$
$$b = \bar{\phi}_2 \cdot \bar{f} = \int_{-\infty}^{\infty} \phi_2(t)\, f(t)\, dt$$

4 - - 71

4-2)





$$\bar{n} = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}, \quad p_n(n_1 n_2) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{(n_1^2+n_2^2)}{2\sigma^2}}$$

$$p_n(n_1 n_2) = \frac{1}{N_0\pi} e^{-\frac{(n_1^2+n_2^2)}{N_0}}$$

$$= \frac{1}{\sqrt{N_0\pi}} e^{-\frac{n_1^2}{N_0}} \frac{1}{\sqrt{N_0\pi}} e^{-\frac{n_2^2}{N_0}}$$

$$= p_n(n_1)\, p_n(n_2)$$

WE CAN DEFINE 2 ORTHONORMAL TIME FUNCTIONS, $\phi_1(t), \phi_2(t)$, THEN THE SIGNALS MAY BE WRITTEN:

$$\begin{cases} s_0(t) = \sqrt{E_s}\left(\frac{1}{\sqrt{2}}\phi_1(t) + \frac{1}{\sqrt{2}}\phi_2(t)\right) \\ s_1(t) = \sqrt{E_s}\left(\frac{1}{\sqrt{2}}\phi_1(t) + \frac{1}{\sqrt{2}}\phi_2(t)\right) \\ s_2(t) = -s_0(t) \\ s_3(t) = -s_1(T) \end{cases}$$

$$s_0 = \begin{bmatrix} \sqrt{\frac{E_s}{2}} \\ \sqrt{\frac{E_s}{2}} \end{bmatrix}; \; s_1 = \begin{bmatrix} \sqrt{\frac{E_s}{2}} \\ \sqrt{\frac{E_s}{2}} \end{bmatrix}, \; s_2 = \begin{bmatrix} -\sqrt{\frac{E_s}{2}} \\ -\sqrt{\frac{E_s}{2}} \end{bmatrix}, \; s_3 = \begin{bmatrix} \sqrt{\frac{E_s}{2}} \\ -\sqrt{\frac{E_s}{2}} \end{bmatrix}$$

HOW DO WE MAKE $n(t)$ TO A VECTOR

$$\left.\begin{aligned} n_1 &= \int_{-\infty}^{\infty} n_1(t)\phi_1(\tau)\,d\tau \\ n_2 &= \int_{-\infty}^{\infty} n_2(\tau)\phi_2(\tau)\,d\tau \end{aligned}\right\} \text{PROJECTIONS}$$

(ϕ_1 & ϕ_2 ORTHONORMAL)

n_1 AND n_2 ARE GAUSSIAN R.V.

$P[m_i] = P[m_j] \quad i, j = 0, 1, 2, 3$

MUST MAXIMIZE APOSTERORI PROBIBILITY:

$P[m_i]\,p_r(\bar{\rho}\,|\,s=s_i)$; i.e.

$\hat{m}(\rho) = m_i$ WHEN

$P[m_i]\,p_r(\rho\,|\,s=s_i) > P[m_k]\,p_r(\rho\,|\,\bar{s}=s_k)$ $i \neq k$

BECAUSE $P[m_i] = 1/4$, WE HAVE MAXIMUM LIKLIHOOD SITUATION AND THE DECISION FUNC. BECOMES:

$p_r(\rho\,|\,s=s_i) \qquad (\bar{n} = \bar{r} - \bar{s}_i)$

NOW $p_r(\bar{\rho}\,|\,s=\bar{s}_k) = p_n(\bar{\rho} - \bar{s}_k)$

$$= \frac{1}{\pi N_0}\, e^{-\left[(\rho_1 - s_{k1})^2 + (\rho_2 - s_{k2})^2\right]/N_0}$$

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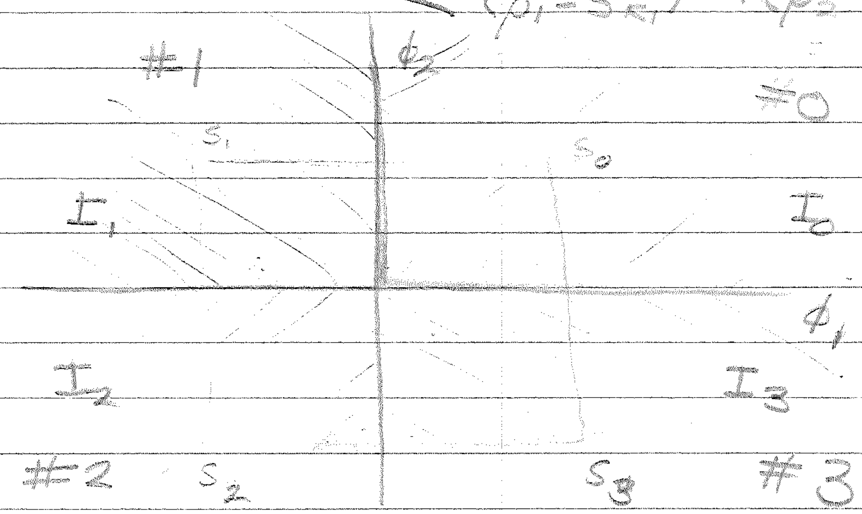
AND $\frac{1}{\pi N} e^{-[(\rho_1 - s_{i1})^2 + (\rho_2 - s_{i2})^2]/N_0}$

$> \frac{1}{\pi N} e^{-[(\rho_1 - s_{k1})^2 + (\rho_2 - s_{k2})^2]/N_0} \quad k \neq i$

IFF $(\rho_1 - s_{i1})^2 + (\rho_2 - s_{i2})^2$

$< (\rho_1 - s_{k1})^2 + (\rho_2 - s_{k2})^2$



$p[C/m_i] = p[r \in I_i / m_i]$

$= \iint_{I_i} p_r(\rho / s = s_i) \, d\rho$

FOR EXAMPLE:

$p[C/m_0] = \int_0^\infty \int_0^\infty \frac{1}{\pi N_0} e^{-[(\rho_1 - s_{01})^2 + (\rho_2 - s_{02})^2]/N_0} \, d\rho_1 \, d\rho_2$

$= \int_0^\infty \int \frac{1}{\pi N_0} e^{-[(\rho_1 - \sqrt{\frac{E}{2}})^2 + (\rho_2 - \sqrt{\frac{E}{2}})^2]/N_0} \, d\rho_1 d\rho_2$

$= \left(\frac{1}{\pi N_0}\right)^{\frac{1}{2}} \int_0^\infty e^{-\frac{(\rho_1 - \sqrt{E/2})^2}{N_0}} \, d\rho_1$

$\left(\frac{1}{\pi N_0}\right)^{\frac{1}{2}} \int_0^\infty e^{-\frac{(\rho_2 - \sqrt{\;})^2}{N_0}} \, d\rho_2$

$= \left[\left(\frac{1}{\pi N_0}\right)^{\frac{1}{2}} \int_0^\infty e^{-\frac{(\rho - \sqrt{\;})^2}{N_0}}\right) d\rho$

$= Q^2\left(-\sqrt{\frac{E_s}{N_0}}\right)$

$\Rightarrow Q(-\sqrt{\;}) = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{\;}}^\infty e^{-\ell^2/2} \, d\ell$

$\ell/2 = (\rho - \sqrt{\;})^2 \sqrt{2/N_0}$

VECTOR LENGTH = ENERGY

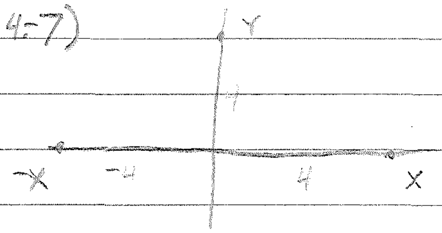
$\Rightarrow P[c/m_0] = Q^2\left(-\sqrt{\frac{E_s}{N_0}}\right)$

$\therefore P[c/m_i) = \quad '' \qquad \ni i = 0,1,2,3$

$\Rightarrow P[c] = Q^2\left(-\sqrt{\frac{E_s}{N_0}}\right)$

$\frac{E_s}{N_0}$ = SIGNAL TO NOISE RATIO

4-7)



X AND -X BETTER
THAN X AND Y
(FURTHER APART)

DEFINE AN ORTHONORMAL SET

$\phi_1(t) = \frac{1}{4}x(t)$

$\phi_2(t) = \frac{1}{4}y(t)$

THEN $\int_{-\infty}^{\infty}\phi_1(t)\phi_2(t)dt = 0$

$\int_{-\infty}^{\infty}\phi_i^2(t)dt = 1 \qquad i = 1,2$

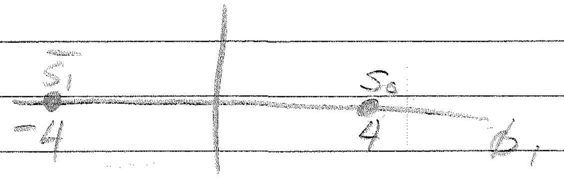
SPECTRAL DENSITY OF NOISE:

$S(f) = \frac{N_0}{2} = 4 \ \frac{\text{WATT-SEC}}{\text{CYCLE}} = 4 \text{ joule}$

$\Rightarrow N_0 = 8$

a) LET $s_0(t) = 4\phi_1(t) \Rightarrow \bar{s}_0 = 4\phi_1$

$s_1(t) = -4\phi_1(t) \Rightarrow \bar{s}_1 = -4\phi_1$



$P[m_0] = P[m_1] = \frac{1}{2}$

$\therefore \hat{m}(\rho) = m_0$ IFF

$p_n(\rho - s_0) > p_n(\rho - s_i)$

IN THE FIRST PROBLEM,

ALL VECTORS ARE ONE DIMENSIONAL

BY DEFINITION;

$\bar{n} = n = \int_{-\infty}^{\infty} n_w(t)\,\phi_1(t)\,dt$ (4D)

$E[n] = \int_{-\infty}^{\infty} E[n_w(t)]\,\phi_1(t)\,dt = 0$

$n^2 = \int_{-\infty}^{\infty} n_w(t_1)\,\phi_1(t_1)\,dt_1 \int_{-\infty}^{\infty} n_w(t_2)\,\phi_1(t_2)\,dt_2$

$= \iint_{-\infty}^{\infty} n_w(t_1)\,n_w(t_2)\,\phi_1(t_1)\,\phi_1(t_2)\,dt_1\,dt_2$

$$E[n^2] = \iint_{-\infty}^{\infty} \underbrace{E[n_w(t_1)\,n_w(t_2)]}_{\text{AUTOCORRELATION OF WHITE NOISE}}\,\phi_1(t_1)\,\phi_2(t_2)\,dt_1\,dt_2$$

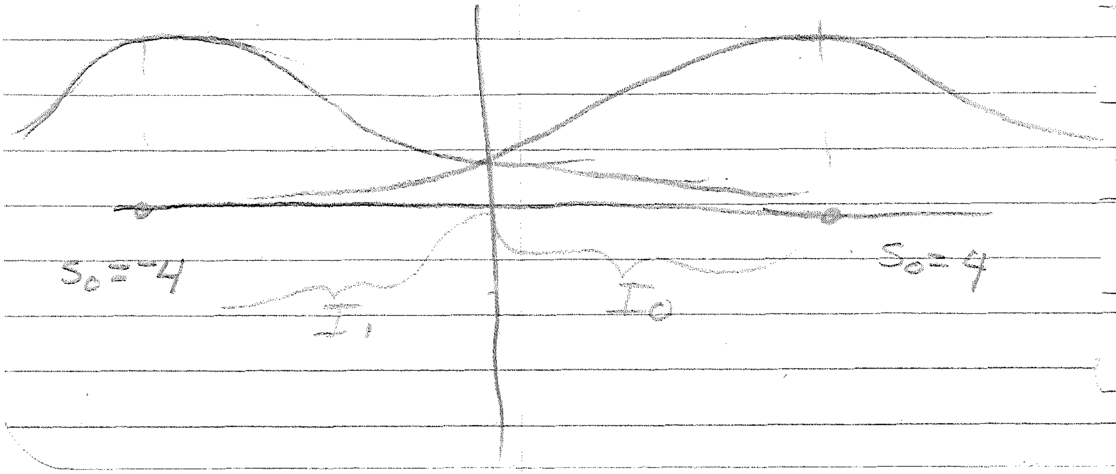
$$= \frac{N_0}{2}\iint_{-\infty}^{\infty} \delta(t_1 - t_2)\,\phi_1(t_1)\,\phi_1(t_2)\,dt_1\,dt_2$$

$$= \frac{N_0}{2}\int_{-\infty}^{\infty} \phi_1(\tau_1)\,\phi_1(t_1)\,d\tau_1 = \frac{N_0}{2}$$

$E[n^2] = \sigma^2$ (FOR 0 MEAN) $= N_0/2$

$$\therefore p_n(\bar{\rho} - \bar{s}_0) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(\rho - s_0)^2}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{\pi N_0}}\, e^{-\frac{(\rho - s_0)^2}{N_0}} = \frac{1}{2(2\pi)^{1/2}}\, e^{-\frac{(\rho - s_i)^2}{8}}$$



$$P[C/m_0] = \int_0^{\infty} \frac{1}{2\sqrt{2\pi}} e^{-\frac{(\rho - s_0)^2}{8}} d\rho$$

LET $\hat{\phi} = \frac{\rho - s_0}{2}$; $d\hat{\phi} = \frac{1}{2} d\rho$

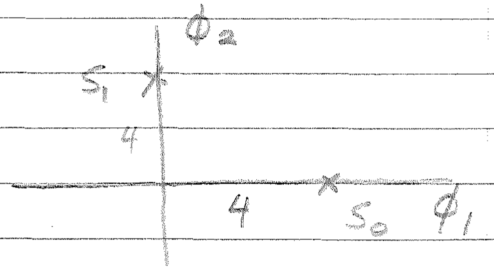
$$\Rightarrow P[C/m_0] = \frac{1}{\sqrt{2\pi}} \int_{-2}^{\infty} e^{-\hat{\phi}^2/2} d\hat{\phi}$$

$$= Q(-2)$$

$$P[C/m_1] = P[C/m_0] = Q(-2)$$

$$\Rightarrow P[C] = Q(-2)$$

b)



$s_0 = [4, 0]$; $s_1 = [0, 4]$

$\bar{n} = [n_1, n_2]$

$n_1 = \int_{-\infty}^{\infty} n_w(t) \phi_1(t) dt$

$n_2 = \int_{-\infty}^{\infty} n_w(t) \phi_2(t) dt$

$E[n_1] = E[n_2] = 0$

$E[n_1^2] = E[n_2^2] = N_0/2$

$E[n_1 n_2] = \iint E[n_w(t_1) n_w(t_2)] \phi_1(t_1) \phi_2(t_2) dt_1 dt_2$

$= \frac{N_0}{2} \int^{\infty} \phi_1(t_1) \phi_2(t_1) dt_1 = 0$

∴ NOISE COMPONENTS INDEPENDENT FOR THE PROJECTIONS ONTO TWO ORTHOGONAL FUNC. $\phi_1(t)$, $\phi_2(t)$

$$\therefore p_n(\alpha) = \frac{1}{2\pi \frac{N_0}{2}} e^{(\alpha_1^2 + \alpha_2^2)/2 N_0/2}$$

(CONT)

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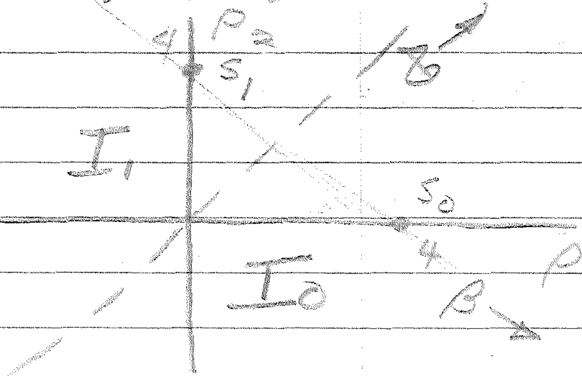
$$p_n(\rho - s_0) = \frac{1}{2\pi \frac{N_0}{2}} e^{-\left[\frac{(\rho_1-4)^2+\rho_2^2}{2N_0/2}\right]}$$

$$p_n(\rho - s_1) = \frac{1}{\pi N_0} e^{-\left[(\rho_1^2+(\rho_2-4)^2)/N_0\right]}$$



$$P[c/m_0] = \int_{-\infty}^{\infty} d\rho_2 \int_{\rho_2}^{\infty} d\rho_1 \frac{1}{\pi N_0} e^{-[(\rho_1-4)^2+\rho_2^2]/N_0}$$

LET $\mathcal{B} = \rho_1 - 4$

$$P[c/m_0] = \int_{-\infty}^{\infty} d\rho_2 \int_{\rho_2-4}^{\infty} d\mathcal{B} \frac{1}{\pi N_0} e^{-\frac{(\mathcal{B}^2+\rho_2^2)}{N_0}}$$

LET $\alpha = \frac{1}{\sqrt{2}}(\mathcal{B} - \rho_2)$

$\beta = \frac{1}{\sqrt{2}}(\mathcal{B} + \rho_2)$

THEN

$$P[c/m_0] = \int_{-\infty}^{\infty} d\beta \int_{-2\sqrt{2}}^{\infty} d\alpha \frac{1}{\pi N_0} e^{-(\alpha^2+\beta^2)/N_0}$$

$$= \frac{1}{\sqrt{\pi N_0}} \int_{-\infty}^{\infty} d\alpha\, e^{-\frac{\alpha^2}{N_0}} = Q(-\sqrt{2})$$

$$P[c/m_1] = Q(-\sqrt{2})$$

$$\Rightarrow P[c] = Q[-\sqrt{2}] < Q[-2]$$

5-13-71

RECEIVER IMPLEMENTATION

M-MESSAGE, $i = 0, \ldots, M-1$

N-DIMENSIONS $j = 1, \ldots, N$

RECEIVER CALCULATES:

$$\bar{r} = (r_1, \ldots, \bar{r}_2)$$

$$r_j = \int_{-\infty}^{\infty} r(t)\,\phi_j(t)\,dt$$

= PROJECTION OF $r(t)$ ONTO $\phi_j(t)$

DECISION FUNCTION

$$|\bar{r} - s_i|^2 - N_0 \ln P[m_i]$$

FOR W.G.N, $S_n f = \frac{N_0}{2}$; $\sigma^2 = \frac{N_0}{2}$

$\hat{m} = m_k$ IF DECISION FUNCTION IS MINIMUM FOR $i = k$

NOW $|r - s_i|^2 = \sum_{j=1}^{N} (r_j - s_{ij})^2 = |\bar{r}|^2 - 2 r_1 s_i + |s_i|^2$

$\Rightarrow \bar{r}_1 \bar{s}_i = \sum_{j=1}^{N} r_j s_{ij}$

HENCE, BECAUSE OF THE (−) SIGN BEFORE $\bar{r} \cdot s_i$, MUST MAXIMIZE $\bar{r} \cdot s_i$ IN ORDER TO MINIMIZE $|\bar{r} - s_i|^2$

REALLY WE MUST MAXIMIZE.

$$\bar{r} \cdot s_i + c_i \ni c_i = \tfrac{1}{2}(N_0 \ln P[m_i] - |s_i|^2)$$

$$i = 0, \ldots, M-1$$

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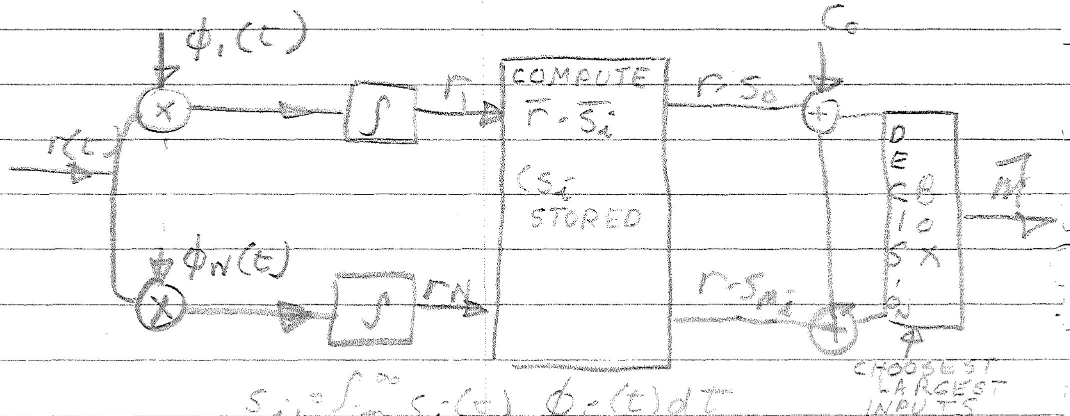
CORRELATION RECEIVER COMPUTES THE (CROSS) CORRELATION FUNCTION $\overline{r}\cdot\overline{s}_i$, EITHER ANALOGICALLY OR DIGITLY

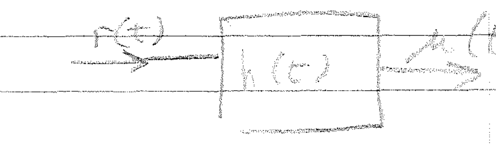
NOTE $\overline{r}\cdot\overline{s}_i = \int_{-\infty}^{\infty} r(t)\, s_i(t)\, dt$

OR $r_i = \int_{-\infty}^{\infty} r(t)\, \phi_i(t)\, dt$



$s_{ij} = \int_{-\infty}^{\infty} s_i(t)\, \phi_j(t)\, dt$

MATCHED FILTERS (p. 236)



$\mu(t) = \int_{-\infty}^{\infty} r(\alpha)\, h(t-\alpha)\, d\alpha$

WE WANT TO GENERATE THE COMPONENTS, r_i, OF THE RMC

$r_i = \int_{-\infty}^{\infty} r(\alpha)\, \phi_i(\alpha)\, d\alpha$

CONSIDER;

$h(t) = \phi_i(T-t);\ i = N$

$\mu(t) = \int_{-\infty}^{\infty} r_\alpha (T-(t-\alpha))\, d\alpha$

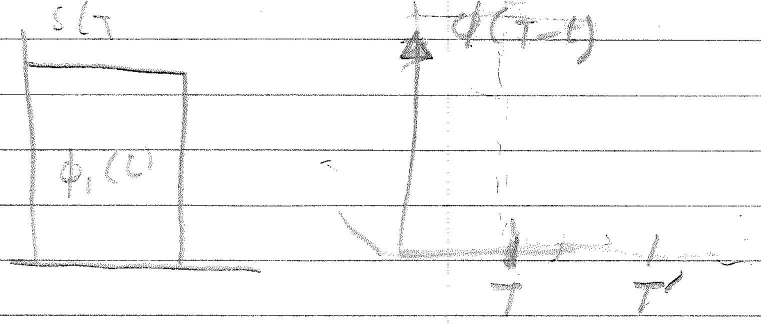
$= \int_{-\infty}^{\infty} r(\alpha)\phi(T+\alpha-t)d\alpha$

SAMPLE AT $t = 0$ AND GET

$y(t) = r_j = \int_{-\infty}^{\infty} r(\alpha)(\phi_1(\alpha)d\alpha$

h(t) MUST BE CAUSAL

$h(t) = 0$; SUPER IMPULSE



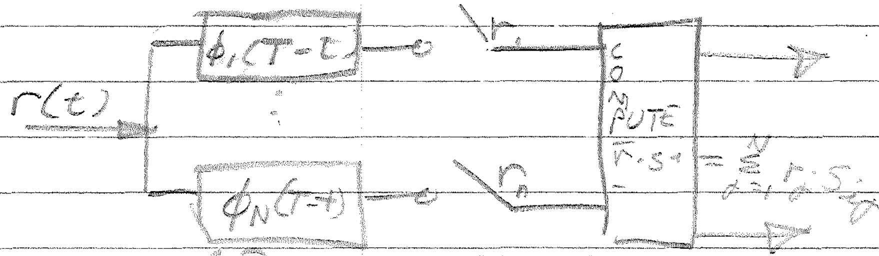
FOR CAUSALITY ON $\phi(T-t)$, WE FORGET ~~THAT~~, WE FORGIVE THAT EACH BASIC FUNC. VANISH IDENTICALLY OUTSIDE SOME FINITE INTERVAL [illegible] T

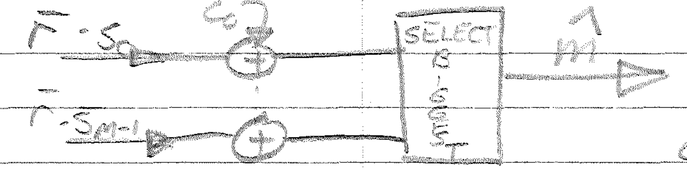
CAN MATCH DIRECTLY TO $S_i(t)$

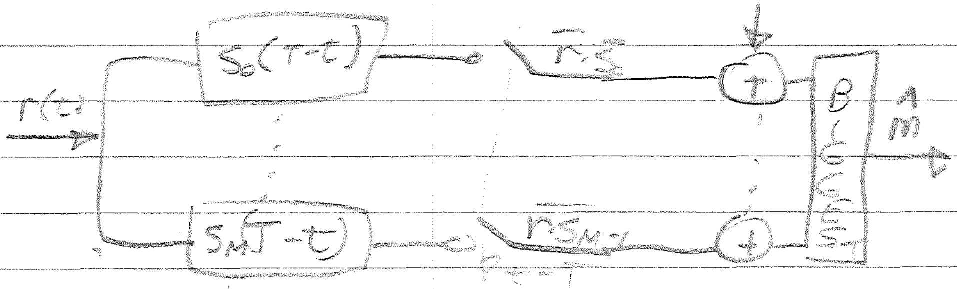
$$u(t) = \int_{-\infty}^{\infty} r(\alpha)\, S_i(T-(t-\alpha))\, d\alpha$$

$$= \int_{-\infty}^{\infty} r(\alpha)\, S_i(T+\alpha-t)\, d\alpha$$

$$\therefore\ u(T) = \bar{r}\cdot\bar{s}_i = \int_{-\infty}^{\infty} r(\alpha)\, S_i(\alpha)\, d\alpha$$



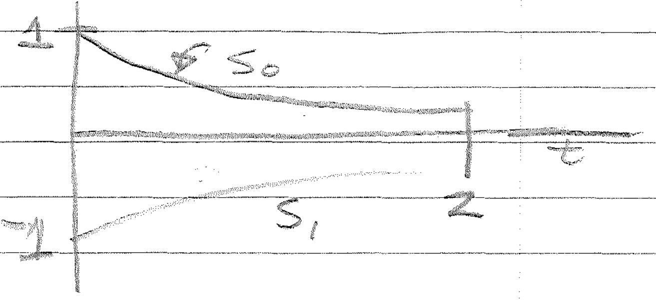




MATCHING DIRECTLY TO SIGNAL

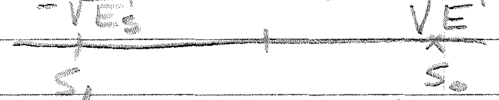
4-15)

$s_0 = e^{-t}\mu(t) \,;\, s_1(t) = -e^{-t}\mu(t)$



ASSUMING $S_N(f) = {}^{N_0}/_2$

$P[E] = Q\left[\frac{d}{\sqrt{2N_0}}\right] \ni d =$ DISTANCE BETWEEN SIGNALS



$E_s = \int_0^\infty e^{-2t}dt = \frac{1}{2}$

$\Rightarrow d = 2\sqrt{E_s}$

$\therefore\ P[E] = Q\left[\frac{2\sqrt{E_s}}{\sqrt{2N_0}}\right] = \boxed{Q\sqrt{\frac{2E_s}{N_0}}}$

E_s (REALLY) $= \int_0^2 e^{-2t}dt = \frac{1}{2}(1-e^{-4})$

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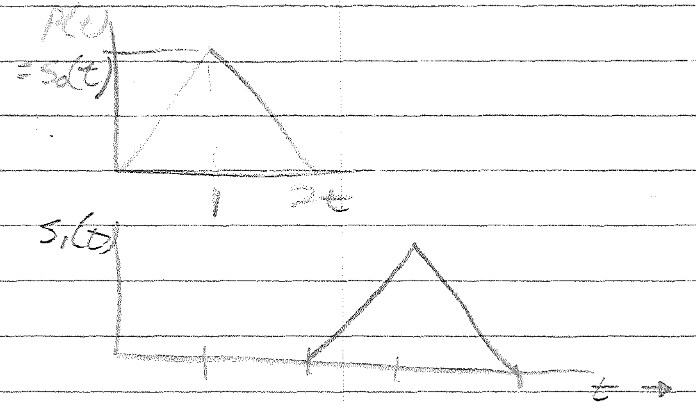
4-10)



a) AN OPTIMUM RECEIVER
SELECTS m_0 AFTER
RECEIVING $r(t)$

$$\int_{-\infty}^{\infty} r(t)s_0(t)dt > \int_{-\infty}^{\infty} r(t)s_1(t)dt$$

SINCE THE ENERGY OF
$s_0(t)$ = ENERGY IN $s_1(t)$
AND BOTH ARE EQUALLY
PROBABLE
($C_0 = C_1$, HENCE, IGNORE)

b) FOR EQUALLY LIKELY BINARY
MESSAGES:

$$P[\varepsilon] = Q\left[\frac{d}{\sqrt{2N_0}}\right]$$

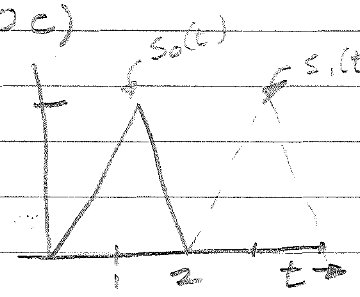
WHERE $d^2 = |s_0 - s_1|^2$

$$= \int_{-\infty}^{\infty} [s_0(t) - s_1(t)]^2 dt = \frac{16}{3}$$

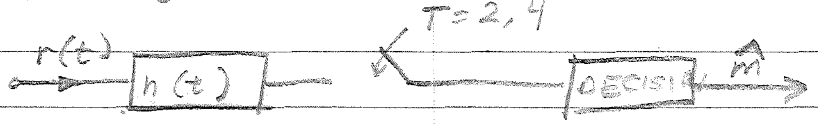
$$\Rightarrow P[E] = Q\left[\frac{4}{\sqrt{6}(.1)}\right] \approx 10^{-7}$$

4-17-71

4-10 c)



c) METHOD 1

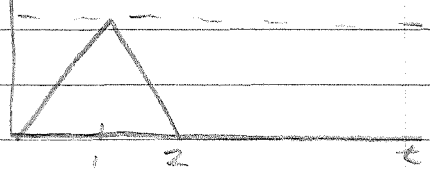


THE SAMPLE AT T = 2 IS

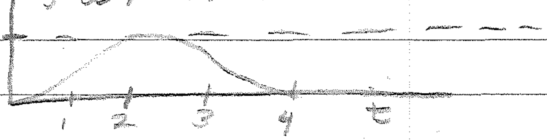
$\int_{-\infty}^{\infty} r(t) s_0(t) dt$,

THAT AT T = 4 IS $\int_{-\infty}^{\infty} r(t) s_1(t) dt$

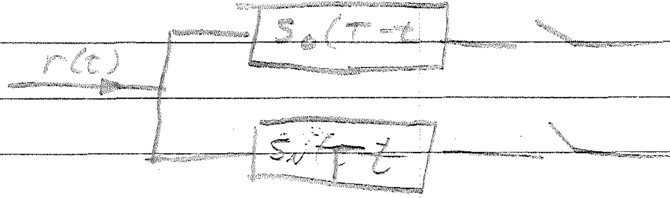
$h(t) = s_0(2-t)$

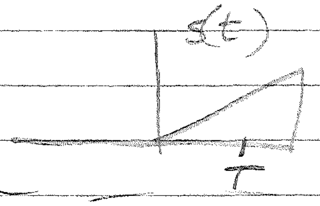


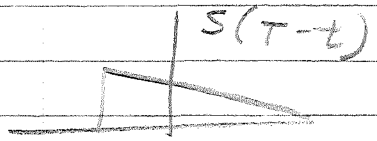
$\int_{-\infty}^{\infty} r(\tau) h(t-\tau) d\tau$



MATCH FILTER



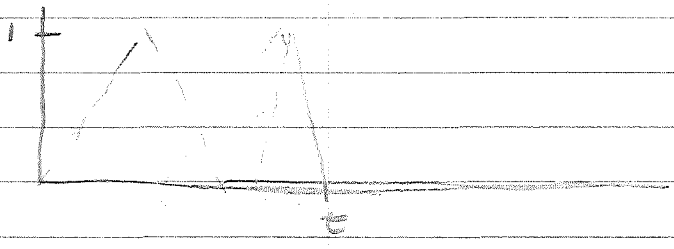




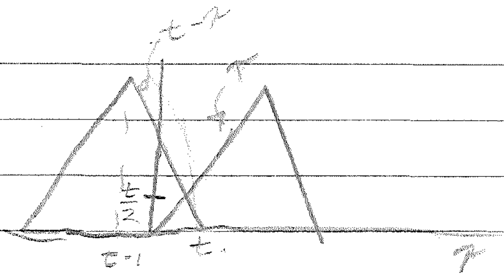
(CONT)

$$\int_{-\infty}^{\infty} r(\alpha)\, h(t-\alpha)\, d\alpha = e(t)$$
$$= \int_{-\infty}^{\infty} r(\alpha)\, s_0(T-(t-\alpha))\, d\alpha$$
$$= \int_{-\infty}^{\infty} r(\alpha)\, s_0(T-t+\alpha)\, d\alpha$$
$$v(2) = \int_{-\infty}^{\infty} r(\alpha)\, s_0(\alpha)\, d\alpha$$
$$v(4) = \int_{-\infty}^{\infty} r(\alpha)\, s_0(\alpha-2)\, d\alpha$$
$$= \int_{-\infty}^{\infty} r(\alpha)\, s_1(\alpha)\, d\alpha$$

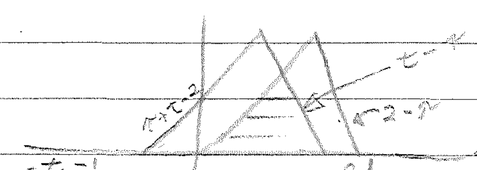
$$\int_{-\infty}^{\infty} p(\tau)\, s_0(t-\tau)\, d\tau$$
$$= \int_{-\infty}^{\infty} p(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} p(\tau)\, s_0(2-t+\tau)\, d\tau$$
$$= \int_{-\infty}^{\infty} s_0(\tau)\, s_0(2-t+\tau)\, d\tau$$
$$= \int_{-\infty}^{\infty} s_0(\tau)\, s_0(t-\tau)\, d\tau$$



$t < 0 \Rightarrow * = 0$

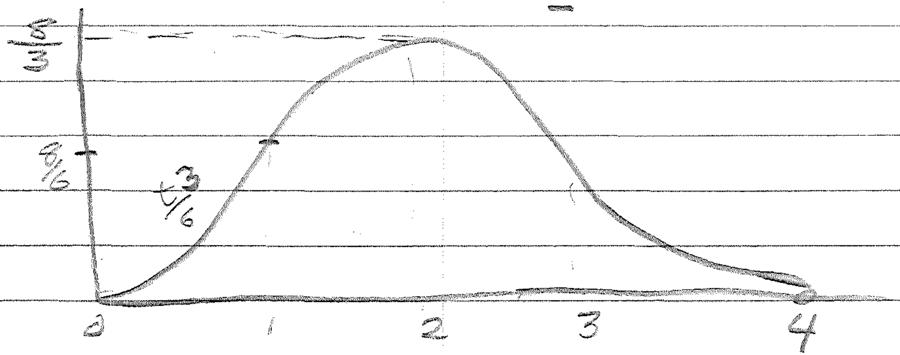


$\int_0^t (t-\tau)\tau\, d\tau = -\frac{t^3}{3} + \frac{t^3}{2} = \frac{t^3}{6} \quad 0<t<1$

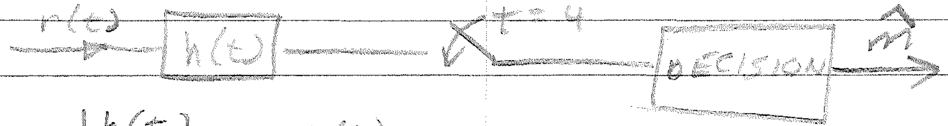


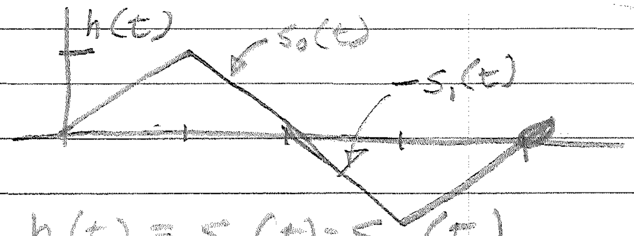
$\int_0^{t-1} \tau(\tau+t-2) + \int_{t-1}^{1} \tau(t-\tau)d\tau + \int_1^t (t-\tau)(2-\tau)d\tau$

=



FOR METHOD II





$h(t) = s_0(t) - s_1(t)$

(cont)

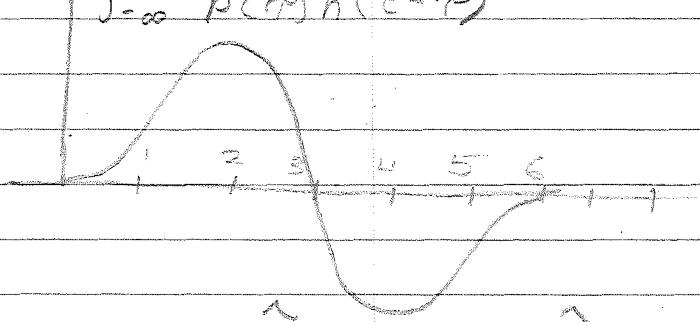
③③

WRONG

$\int_{-\infty}^{\infty} r(t)h(\tau-\tau)dt$

$= \int_{-\infty}^{\infty} r(\tau)\left[s_0(t)-s_1(\tau)\right]dt$

$\int_{-\infty}^{\infty} p(\tau)h(t-\tau)$



FOR +, $\hat{m} = S_0$ $\hat{m} = s_1$ FOR −

$h(t) = s_1(4-t) - s_0(4-t)$

$\int_{-\infty}^{\infty} r(\tau)\left[s_1(4+\tau-t - S_0(4+\tau-t)d\tau\right.$

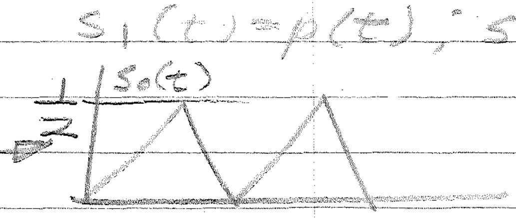
(RIGHT GRAPH)

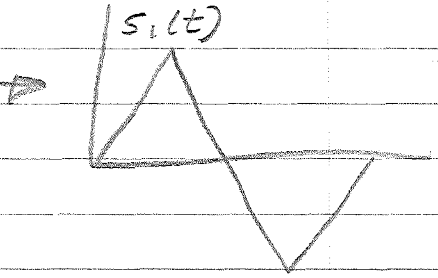
d) SAME ERROR PROB

$S_0(t) = \frac{1}{2}\left[p(t)+p(t-2)\right]$

SMALL ERROR (ENTIPITAL

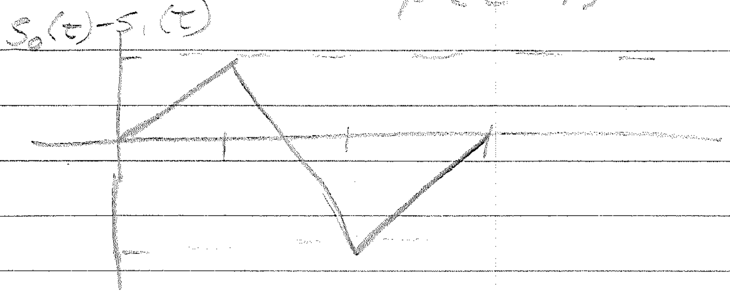
$s_1(t) = p(t)$; $s_0(t) = -p(t)$





e) FOR $S_0(t) = p(t)$

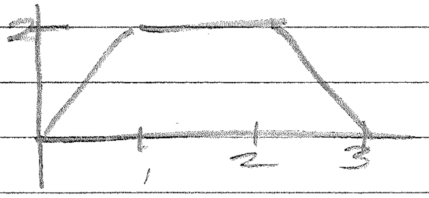
$S_1(t) = p(t-1)$



$$d^2 = \int_{-\infty}^{\infty} [S_0(t) - S_1(t)]^2 dt$$

$$= 2\int_0^1 (2x)^2 dx + 2\int_0^{\frac{1}{3}} (4x)^2 dx = 4$$

$$P[\varepsilon] = Q\left[\frac{1}{\sqrt{.05}}\right]$$



FOR $S_0(t) = p(t)$, $S_1(t) = -p(t-1)$

$$d^2 = \int_{-\infty}^{\infty} [S_0(t) - S_1(t)]^2 dt = \frac{20}{3}$$

$$P[E] = Q\left[\sqrt{\frac{5}{3(.05)}}\right] \cong 4 \times 10^{-9}$$

4-21-71

$$\int_{-\infty}^{\infty} f(t)\, g(t)\, dt = \int_{-\infty}^{\infty} F(f)\, G(f)\, df$$

$$d^2 = \int_{-\infty}^{\infty} [s_1(t) - s_0(t)]^2\, dt$$
$$= \int_{-\infty}^{\infty} (s_1^2(f) - 2 S_1(f) S_0(f) + S_0^2(f))\, df$$

FOR REAL TIME FUNCTIONS:

$$S_0(-f) = S_0^*(f)$$

CHAPT. 7 : FIRST 2 SECTIONS

COMPLEX CONVOLUTION

$$\mathcal{F}\{g(t)h(t)\} = \int_{-\infty}^{\infty} G(\beta) H^*(f-\beta)\, d\beta$$
$$G(f) = \mathcal{F}(g(t))$$
$$H(f) = \mathcal{F}(h(t))$$

$$\mathcal{F}(g(t)h(t)) = \int_{-\infty}^{\infty} e^{-j2\pi f t}\, g(t)\, h(t)\, dt$$

$$= \int_{-\infty}^{\infty} e^{-j2\pi f t} \left(\int_{-\infty}^{\infty} G(f_1)\, e^{j2\pi f_1 t}\right.$$
$$\left. df_1\right) \int_{-\infty}^{\infty} H(f_2)\, e^{j2\pi f_2 t}$$

$$= \iint_{-\infty}^{\infty} G(f_1)\, H(f_2)$$
$$\left(\int_{-\infty}^{\infty} e^{-j2\pi(f-f_1-f_2)t}\, dt\right) df_1\, df_2$$

$$\int_{-\infty}^{\infty} e^{-j\pi f t}\, dt \triangleq \delta(f)$$

$$\Rightarrow \mathcal{F}(g(t)h(t)) = \iint H(f_1) \cdot G(f_2)$$
$$\delta(f - f_1 - f_2)\, df_1\, df_2$$
$$= \int_{-\infty}^{\infty} G(\beta)\, H(f-\beta)\, d\beta$$

FRI 4-25-71

7-4) $n(t) = n_c(t)\sqrt{2}\cos\omega_0 t + n_s(t)\sqrt{2}\sin\omega_0 t$

$\overline{n(t)n(t-\tau)} = [n_c(t)\sqrt{2}\cos\omega_0 t + n_s(t)$

$\sqrt{2}\sin\omega_0 t][n_c(t-\tau)\sqrt{2}\cos\omega_0(t-\tau)$

$+ n_s(t-\tau)\sqrt{2}\sin\omega_0(t-\tau]$

$= \overline{n_c(t)n_c(t-\tau)}\,2\cos\omega_0 t\cos\omega_0 t-\tau$

$+ \overline{n_s(t)n_s(t-\tau)}\,2\sin\omega_0 t\sin\omega_0(t-\tau)$

$+ \overline{n_s(t)n_c(t-\tau)}\,2\sin\omega_0 t\cos\omega_0(t-\tau)$

$+ n_c(t)n_s(t-\tau)\,2\cos\omega_0 t\sin\omega_0(t-\tau)$

$= R_c(\tau)[\cos\omega_0 t + \cos(2\omega_0 t-\omega_0\tau))$

$+ R_s(\tau)[\cos\omega_0\tau - \cos(2\omega_0 t-\omega_0\tau)]$

$+ R_{sc}(\tau)[\sin(2\omega_0 t-\omega_0\tau) + \sin\omega_0\tau]$

$+ R_{cs}(\tau)[\sin(2\omega_0 t-\omega_0\tau) - \sin\omega_0\tau]$

WANT 2 MAKE FUNCTION OF τ

$\overline{n(t)n(t-\tau)} = \cos\omega_0\tau[R_c(\tau)+R_s(\tau)]$

$+\cos(2\omega_0 t+\omega_0\tau)[R_c(\tau)-R_s(\tau)]$

$+\sin\omega_0\tau[R_{sc}(\tau)-R_{cs}(\tau)]$

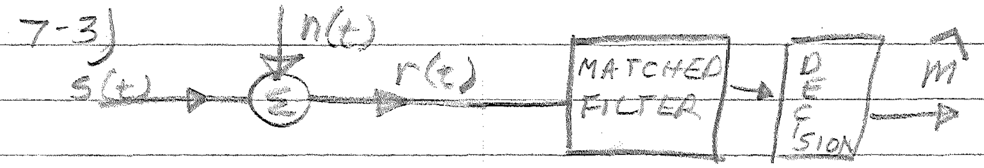
$+\sin(2\omega_0 t-\omega_0\tau)[R_{sc}(\tau)+R_{cs}(\tau)]$

WIDE SENSE STATIONARY SUFFICIENT:

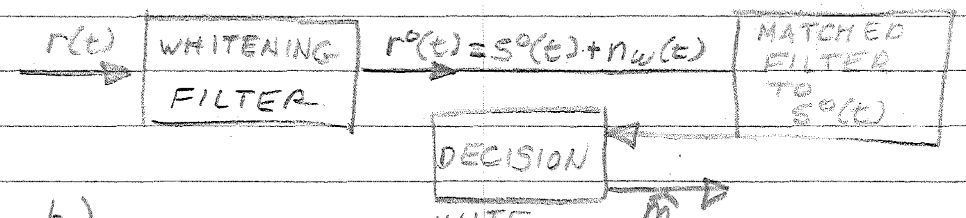
$R_c(\tau)=R_s(\tau)$

$R_{sc}(\tau) = -R_{cs}(\tau)$

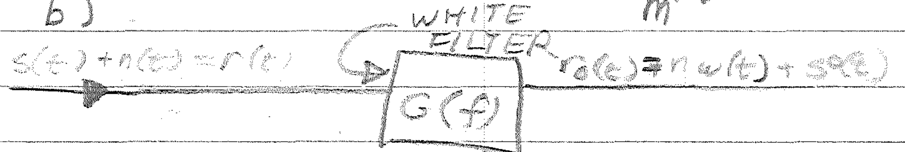
7-3)



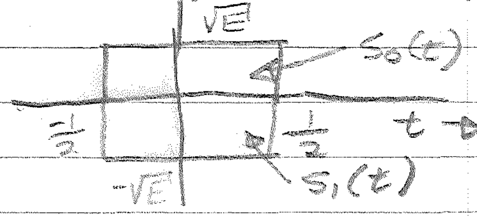
WISH TO PASS r(t) THRU WHITNING FILTER



b)



$$s^o(f) = G(f)S(f)$$



$s_0^o(f) = G(f)S(f)$ (VOLTAGE)

WANNA FIND $G(f)$

$$S_w(f) = \frac{N_0}{2}$$

$$S_n(f) = \frac{N_0}{2} \frac{f^2+1}{f^2+4}$$

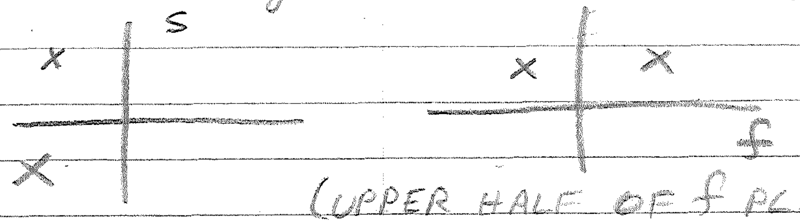
$$S_w(f) = |G(f)|^2 S_n(f)$$ (POWER)

$$\Rightarrow \frac{N_0}{2} = G(f)G^*(f) \frac{N_0}{2} \frac{f^2+1}{f^2+4}$$

$$\Rightarrow G(f)G^*(f) \frac{f^2+1}{f^2+4} = 1$$

$$\Rightarrow G(f)G^*(f) = \frac{1+jf}{2+jf} \frac{1-jf}{1-jf}$$

$$\therefore G(f) = \frac{2+jf}{1+jf}$$ (POLES IN LEFT)



(UPPER HALF OF f PLANE)

$$s = j\pi 2f$$

$$G(s) = \frac{4\pi + j2\pi f}{2\pi + j2\pi f} = \frac{4\pi + s}{2\pi + s}$$

$$\begin{array}{r} 1 + \frac{2\pi}{s+2\pi} \\ s+2\pi \overline{)\, s + 4\pi} \\ \underline{s + 2\pi} \\ 2\pi \end{array} \quad = 1 + \frac{2\pi}{s+2\pi}$$

$\Longrightarrow g(t)$ = IMPULSE RESPONSE FOR WHITENING FILTER

$$g(t) = \mathcal{L}^{-1}[G(t)] = \delta(t) + 2\pi e^{-2\pi t}\mu_{-1}(t)$$

∴ OUTPUT SIGNAL FROM WHITENING FILTER (SIGNAL TO BE MATCHED)

$$s^0(t) = \int_{-\infty}^{\infty} s(\alpha)\, g(t-\alpha)\, d\alpha$$

$$= \pm\sqrt{E_s}\int_{-\frac{1}{2}}^{1/2}\left[\delta(t-\alpha) + e^{-(t-\alpha)}\mu_{-1}(t-\alpha)\right]d\alpha$$

FOR $t < -1/2$, $s_0^0(t) = 0$

FOR $-\frac{1}{2} < t < \frac{1}{2}$

$$s^0(t) = \pm\sqrt{E_s}\left(1 + e^{-t}\int_{-\frac{1}{2}}^{t} e^{\alpha}\, d\alpha\right)$$

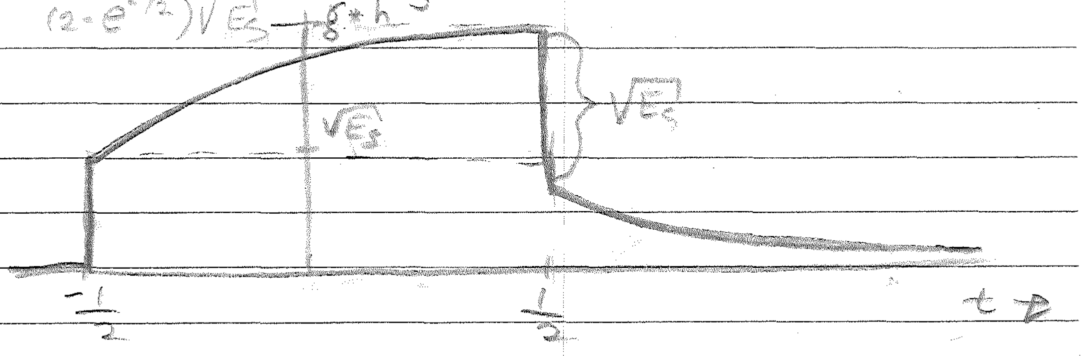
$$= \pm\sqrt{E_s}\left[2 - e^{-(t+\frac{1}{2})}\right]$$

FOR $t > \frac{1}{2}$

$$s^0(t) = \pm\sqrt{E_s}\left[e^{-t}\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{\alpha}\, d\alpha\right]$$

$$= \pm\sqrt{E_s}\left(e^{1/2} - e^{-1/2}\right)e^{-t}$$

$$= \pm\sqrt{E_s}\; 2\sinh\left(\frac{1}{2}\right)e^{-t}$$



THUS, WE CANNOT BUILD A MATCHED FILTER SAMPLING AT TIME T, CAUSE OF THE INFINATE TAIL,

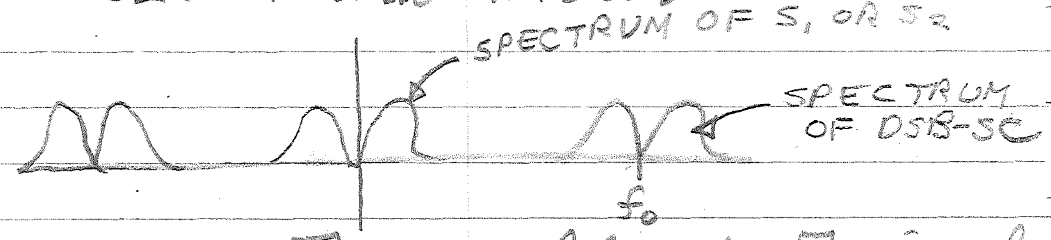
CHOP OFF TAIL IN ABOUT 5 TIME CONSTANTS (5 SEC) WITHOUT TO MUCH LOSS OF OPTIMALITY

MONDAY

7-5) $s_1(t)$, $s_2(t)$ - TWO LOW-PASS (BASE-BAND) SIGNALS

$s_1(t)\sqrt{2}\cos 2\pi f_0 t$

$s_2(t)\sqrt{2}\sin 2\pi f_0(t)$

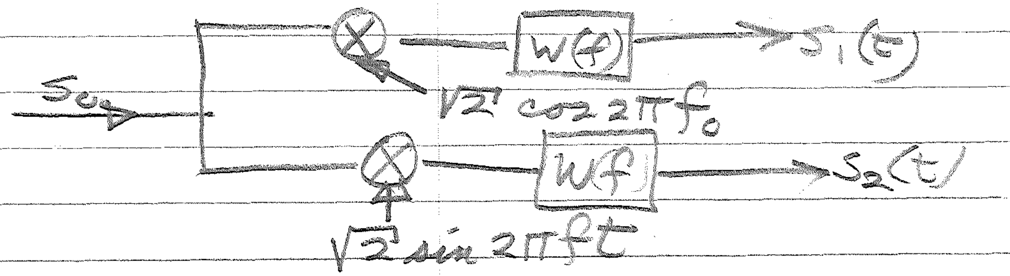


$s_0(t) = s_1(t)\sqrt{2}\cos 2\pi f_0 t + s_2(t)\sqrt{2}\sin 2\pi f_0 t$

(QUADRITURE MULTIPLEXING)

IGNORE CHANNEL EFFECTS

HOW DO WE RECOVER $s_1(t)$ & $s_2(t)$



PROOF

SPECTRUM OF $s_0(t)$:

$$S_0(f) = \frac{1}{\sqrt{2}}\left[S_1(f-f_0) + S_1(f+f_0) - jS_2(f-f_0) + jS_2(f+f_0)\right]$$

$$\mathcal{F}\{s_1(t)\cos 2\pi f_0 t\}$$

$$= \mathcal{F}\{\tfrac{1}{2}s_1(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t})\}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} S_1(\ell)\,\delta(f+f_0-\ell)\,d\ell + \frac{1}{2}\int_{-\infty}^{\infty} S_1(\ell)\,\delta(f+f_0-\ell)\,d\ell \quad \text{(COMPLEX CONVOLUTION)}$$

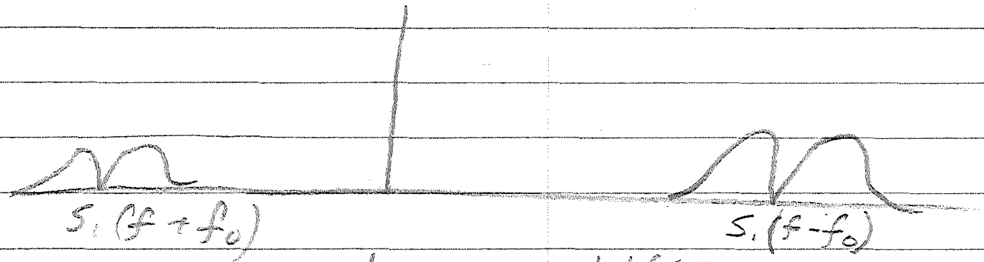
$$\left(\mathcal{F}(e^{j2\pi f_0(t)}) = \int_{-\infty}^{\infty} e^{-j2\pi(f-f_0)t}\,dt = \delta(f-f\right.$$

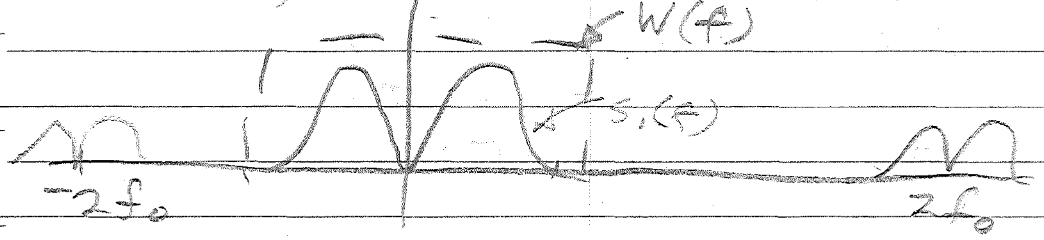
$$\Rightarrow \mathcal{F}\{s_1(t)\cos 2\pi f_0 t\}$$

$$= \frac{1}{2}\left(S_1(f-f_0) + S_1(f+f_0)\right)$$

SHIFT TO RT ↗ ↖ SHIFT LEFT

$$\therefore S_0(f) = \frac{1}{\sqrt{2}}\left[S_1(f-f_0) + S_1(f+f_0) - jS_2(f-f_0) + jS_2(f+f_0)\right]$$



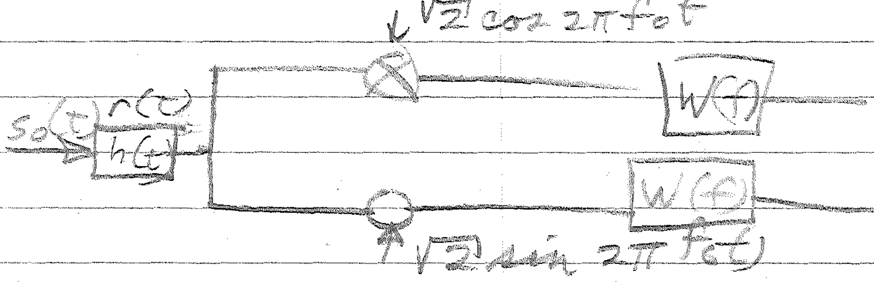


$-j S_2(f-2f_0) + \cancel{j S_2(f)}$

$\cancel{-j S_2(f)} + j S_2(f+f_0)$

NOTHING AT f ORIGIN WHEN MULTIPLIED BY $\cos 2\pi f_0 (t)$

NOW, IF CHANNEL'S IMPULSE IS $R(f) = S_0(f) H(f)$



THE INPUT TO THE LOW-PASS FILTERS

COS CHANNEL

$\sqrt{2}\, r(t) \cos 2\pi f_0 t \rightarrow$ SPECT $\frac{1}{\sqrt{2}} [R(f-f_0)$
$+ R(f+f_0)]$

$= \frac{1}{\sqrt{2}} S_0(f-f_0)\, H\, (f+f_0) + (f+f_0) S_0$
$H(f+f_0)$

$= \frac{1}{2} [H(f-f_0) [S_1(f-2f_0) + S_1(t)$
$- j S_2(f-2f_0) + j S_2(t)]$
$+ H(f+f_0) [S_1(t) + S_1(f+2f_0)$
$- j S_2(f) + j S_2(f+2f_0)]]$

SIN CHANNEL

$$\frac{1}{\sqrt{2}}\left[R(f-f_0) - R[f+f_0)\right]$$

$$= \frac{1}{\sqrt{2}}\left[S_0(f-f_0)\,H(f-f_0)\right.$$

$$\left. - S_0(f+f_0)\,H(f+f_0)\right]$$

$$= \frac{1}{\sqrt{2}}\left\{H(f-f_0)\left[S_1(f-2f_0)\right.\right.$$

$$\left. + S_1(f) - j\,S_2(f-2f_0 + j\,S_2(f)\right]$$

$$- H(f+f_0)\left[S_1(f) + S_1(f+2f_0)\right.$$

$$\left.\left. - j\,S_1(f) + j\,S_2(f+2f_0)\right]\right\}$$

OUTPUT ONLY CONTAINS $S_n(f)$

$\Rightarrow$ OUTPUT OF COS CHANNEL

$$\frac{1}{2}\left\{S_1(f)\right\}\left[H(f-f_0) + H(f+f_0)\right]$$

$$+ j\,S_2(f)\left[H(f-f_0) + H(f+f_0)\right]$$

$$0 \le f \le W$$

$\Rightarrow$ OUTPUT OF SIN CHANNEL

$$\frac{1}{2}\left[S_1(f)\right]\left[H_1(f-f_0) - H(f+f_0)\right]$$

$$+ j\,S_2(f)\left[H(f-f_0) + H(f+f_0)\right]\Big]$$

$$0 \le f \le W$$

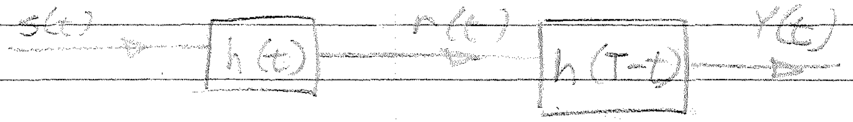
FOR $0 \le f \le W$

$$H(f-f_0) = H(f+f_0)$$

$$H(\omega) = H^*(-\omega)$$

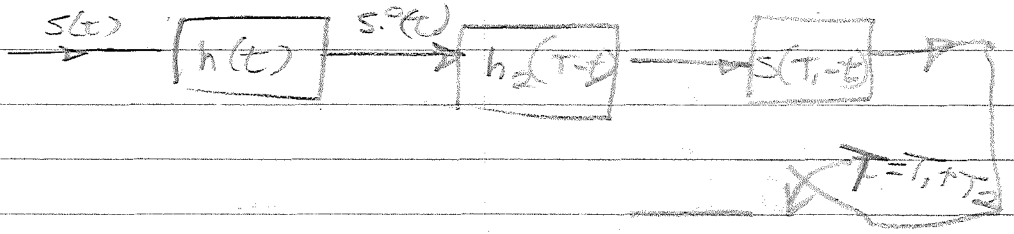
$$\Rightarrow H^*(f_0 - f) = H(f+f_0)$$

FRI.



$$Y(t) = \int_{-\infty}^{\infty} r(\tau_1) h(T+\tau_1 - t) d\tau_1$$

$$R(f) = S(f) H(f)$$
$$Y(f) = R(f) H^*(f) e^{-j2\pi f t}$$
$$= S(f)|H(f)|^2 e^{-j2\pi f t}$$



SEQUENCE OF INPUT PARAMETERS COMMUNICATING THE FOLLOWING RANDOM VECTOR $\overline{m} = (m_1, m_2 \ldots m_K)$

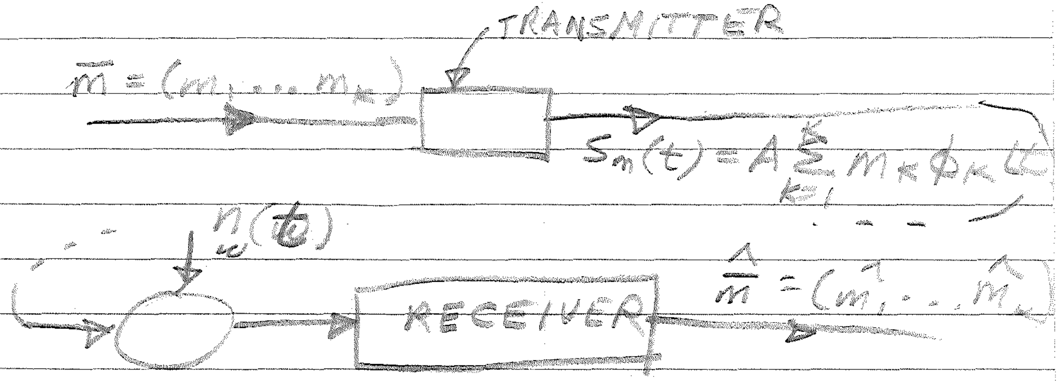
EACH m_i IS CONTINUOUS r.v.

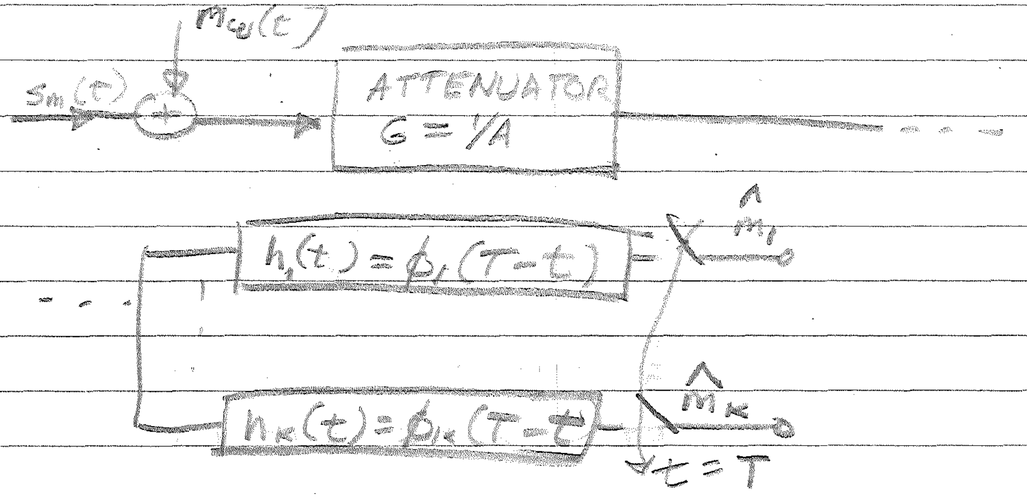
PRIOR TO THIS, ALL MESSAGES WERE DISCRETE OR FINITE IN

LINEAR MODULATION:

$$S_m(t) = \underset{\substack{\uparrow \\ \text{TRANSMITTER GAIN}}}{A} \sum_{m=1}^{K} m_K \overset{\text{ORTHONORMAL}}{\phi_K}(t)$$

OBJECT: ESTIMATE $\hat{\overline{m}} = (\hat{m}_1 \ldots \hat{m}_K)$





INPUT INTO MATCHED FILTER

$$\frac{r(t)}{A} = \sum m_k\phi_k(t) + \frac{n(\omega)}{A}$$

OUTPUT OF FILTER ①

$$= \sum_k m_k \int_{-\infty}^{\infty} \phi_k(t)\phi_1(T+\tau-t)\,d\tau + \frac{1}{A}\int_{-\infty}^{\infty} n_w(\tau)\phi_1(T+\tau-t)\,d\tau$$

$$= \sum m_k \int_{-\infty}^{\infty} \phi_k(\tau)\phi_1(\tau)\,d\tau + \frac{1}{A}\int_{-\infty}^{\infty} n_w(\tau)\phi_1(\tau)\,d\tau = \hat{m}_1$$

$$= m_1 + \frac{1}{A} n_1 \equiv \rho_1/A \ni \rho_1 = Am_1 = n_1$$

= RECEIVED 1ST COMPONENT

ERROR = $\hat{m}_1 - m_1 = \frac{1}{A} n_1$

MEAN SQUARE ERROR
PER COMPONENT
$= E[|\hat{m}_1 - m_1|^2] = E\left[\frac{1}{A^2} n_1^2\right]$
$= \frac{N_0}{2A^2} \Rightarrow N_0/2$ IS
SPECTRAL DENSITY OF
THE WHITE GAUSSIAN NOISE

$$\overline{\varepsilon^2} = \frac{1}{K} \sum_K E[|\hat{m}_K - m_K]^2]$$

$$= \frac{1}{K} E[(\hat{m}_1 - m_1)^2 + \ldots + (\hat{m}_K - m_K)^2]$$

$$= \frac{1}{K} E[|\hat{\overline{m}} - \overline{m}|^2)$$

$$= \frac{1}{K} \frac{K N_0}{2A^2} = \frac{N_0}{2A^2}$$

(AGREES WITH ABOVE)

$$p_m(\alpha) p_r(\rho / m = \alpha)$$

$$= p_{r,m}(\rho, \alpha) = p_m(\alpha | \overset{\uparrow}{r} = \rho) p_r(\rho)$$

IN ORDER THAT THE A POSTERORI PROB. DENSITY $p_m(\alpha | r = \rho)$ BE MAXIMIZED WE MUST

$\frac{p_m(\alpha) p_r(\rho | m = \alpha)}{p_r(\rho)}$ WITH RESPECT TO α

$$\Rightarrow p_m(\alpha)\underbrace{p_r(\rho/m)}_{\text{LIKELIHOOD FUNCTION}} \text{ IS MAX}$$

UPON MAXIMIZING, OUR LIKELIHOOD FUNCTION, WE HAVE MAXIMUN LIKELIHOOD RECEPTION

$$r = n + m$$

$$\Rightarrow p_r(\rho/m=\alpha) = p_n(\rho-\alpha|m=\alpha)$$

NOISE & MESSAGE IND-

$\Rightarrow = p_n(\rho-\alpha)$ WHICH CAN BE MAXIMIZED.

IN K DIMENSIONS

BECAUSE $n_w(t)$ IS WHITE GAUSSIAN NOISE, THE LIKELIHOOD FUNCTION IN K DIMENSIONS IS

$$p_r(\rho/\bar{m}=\alpha) = p_n(\bar{\rho}-\alpha A)$$

$$= \frac{1}{(\pi N_0)^{K/2}} e^{-|\rho-\alpha A|^2/N_0}$$

$$= \prod^{K} \frac{1}{\sqrt{\pi N_0}} e^{-(\rho_k-\alpha_k A)^2/N_0}$$

(NOTE $r_k = Am_k + n_k$)

THE VECTOR $\bar{\alpha}$ THAT MAXIMIZES THIS FUNCTION (FOR A GIVEN VALUE $\bar{\rho}$ OF THE RANDOM VECTOR $\bar{R}$) IS

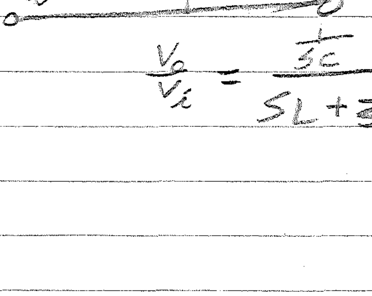
$$\bar{\alpha} = \bar{\rho}/A = \bar{m} + \bar{n}/A = \hat{\bar{m}}$$

THIS IS A LINEAR RECEIVER BECAUE THE ESTIMATE $\hat{\bar{m}}$ OF $\bar{m}$ IS A LINEAR FUNCTION OF THE RECEIVED VECTOR $\bar{\rho}$

RECALL $\hat{\bar{m}} = \bar{m} + \bar{n}/A$ WAS ALSO THE RECEIVER IMPLEMENTED BEFORE

MONDAY

7-2 EX)



$$\frac{V_o}{V_i} = \frac{\frac{1}{SC}}{SL + \frac{1}{SC}} = \frac{1/LC}{S^2 + \omega_o^2}$$

SAMPLING THEOREM: OR HOW TO
REDUCE CONTINUOUS (ANALOG)
DATA TO DISCRETE (DIGITAL)

IF $z(t)$ IS A FINITE ENERGY WAVEFORM WHOSE FORIER TRANSFORM IS IDENTICALLY 0 FOR $|f) \geq W_m$ THEN

$$z(t) = \sum_{k=-\infty}^{\infty} z_k \psi_k(t)$$

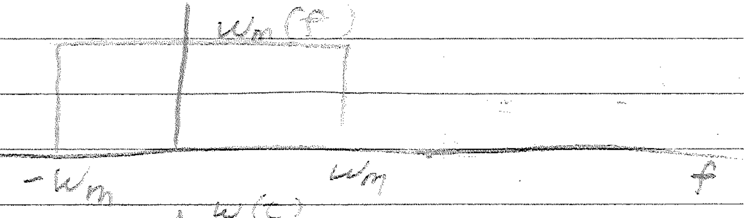
WHERE $z_k = \int_{-\infty}^{\infty} z(t)\,\psi_k(t)\,dt$

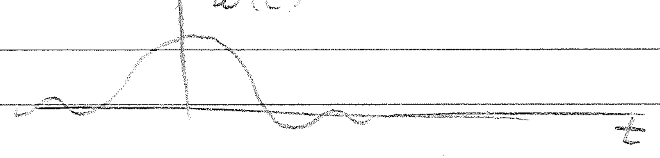
AND $\psi_k(t) = \psi(t - \frac{k}{2W_m})$

$$\psi(t) = \sqrt{2W_m}\,\frac{\sin 2\pi W_m t}{2\pi W_m t}$$

OBERVATIONS:

① IMPULSE RESPONSE OF L.P. FILTER





② $\{\psi_k(t)\}$ ARE ORTHONORMAL

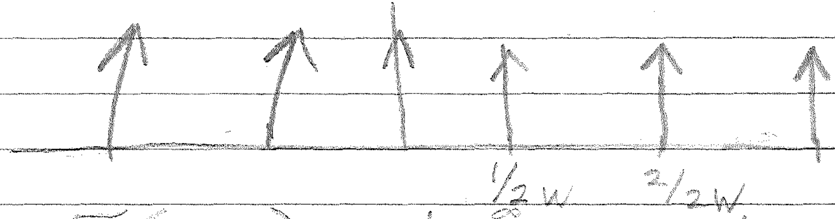
$$\int_{-\infty}^{\infty} \psi_j(t)\,\psi_k(t)\,dt = \int_{-\infty}^{\infty} \Phi_j(f)\,\psi_k(f)\,df$$
$$= 0 \qquad i \neq k$$
$$= 1 \qquad i = k$$
$$= \delta_{ij}$$

97

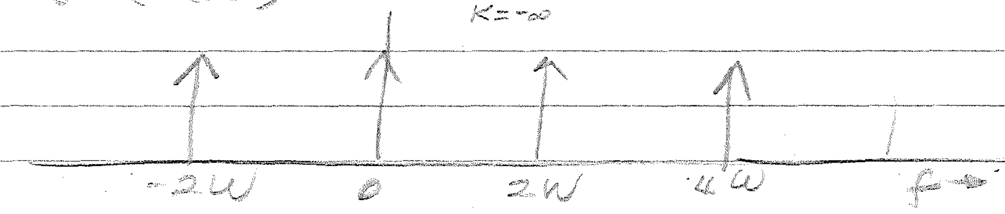
$$z(t) = \sum_{\ell=\infty}^{\infty} K_\ell \int_{-\infty}^{\infty} \psi_\ell(t)\,\psi_k(t)\,dt$$

IMPULSE MODULATION

$$u(t) = \sum_k \delta\left(t - \frac{k}{2W}\right) = \int_{-\infty}^{\infty} U(f)\,e^{j2\pi ft}\,df$$



$$\mathcal{F}(u(t)) = \frac{1}{T} \sum_{K=-\infty}^{\infty} \delta(f - 2kW$$



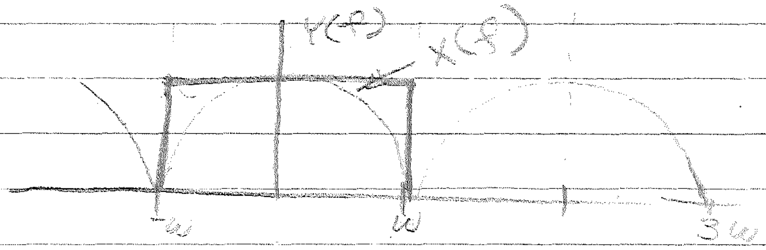
<u>OUTPUT OF IMPULSE MODULATOR</u>

$$v(t) = x(t)\,u(t) = \sum_{K=-\infty}^{\infty} x\left(\frac{k}{2W}\right)\delta(t - k/2W)$$

THEN

$$V(f) = \int_{-\infty}^{\infty} X(\beta)\,U(f-\beta)\,d\beta$$

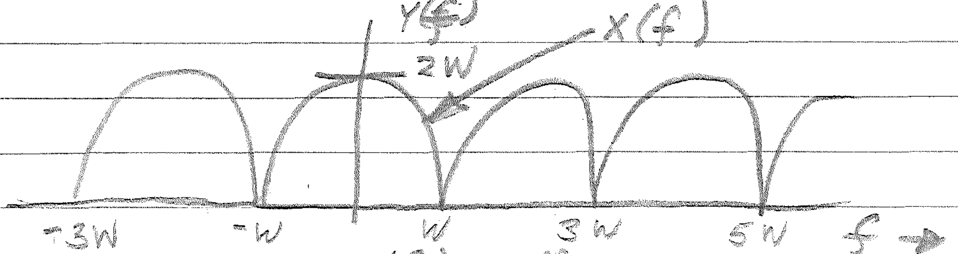
$$= 2W \sum_{K=-\infty}^{\infty} X(f - 2kW)$$

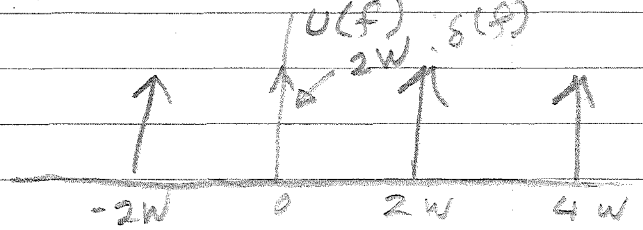


MON : TWO WEEKS FROM TODAY

IMPULSE TRAIN

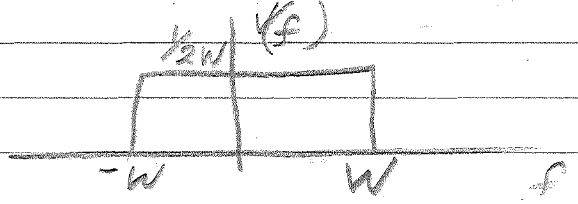
$$Y(t) = X(t)\,\overset{\swarrow}{U}(t) = \sum_{K=-\infty}^{\infty} X(K/2W)\,\delta\left(t-\frac{K}{2W}\right)$$





CAN REGAIN THRU FILTER

$$\Rightarrow X(f) = Y(f)\;V(f)$$



OR $\quad X(t) = Y(t) * V(t)$

$$= \int_{-\infty}^{\infty} Y(\tau)\,V(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} \sum_{K=\infty} X\left(\frac{K}{2W}\right)\delta\left(t-\frac{K}{2W}\right)V(t-\tau)\,d\tau$$

$$= \sum_{K=-\infty}^{\infty} X\left(\frac{K}{2W}\right)\underbrace{V\left(t-\frac{K}{2W}\right)}_{\text{INTERPOLATION FUNCTION}}$$

$$= \sum_{K=-\infty}^{\infty} X\left(\frac{K}{2W}\right)\frac{\sin 2\pi W(t-K/2W)}{2\pi W(t-K/2W)}$$

FIG 8-13 b

RANDOM WAVE FORM: $m(t)$ BANDLIMITED

$$\Rightarrow m(t) = \sum_{K=1}^{K} m_K \psi_K(t)$$

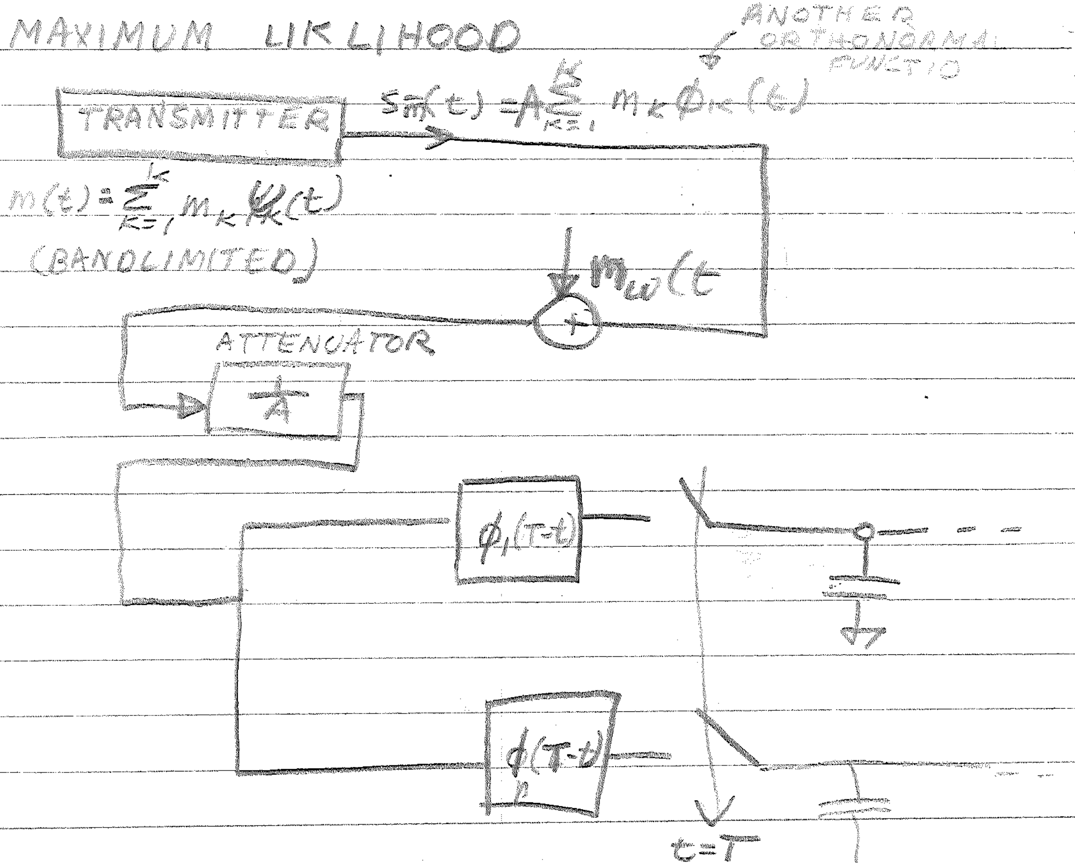
$$\Rightarrow m_K = \int_{-\infty}^{\infty} m(t)\, \psi_n(t) = \frac{1}{\sqrt{2W_M}}\, m\left(\frac{K}{2W_M}\right)$$

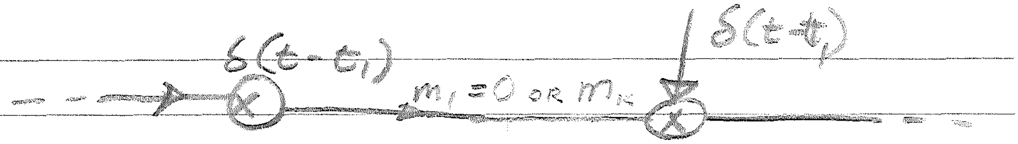
COMMUNICATE, AND THEN ESTIMATE THIS DATA

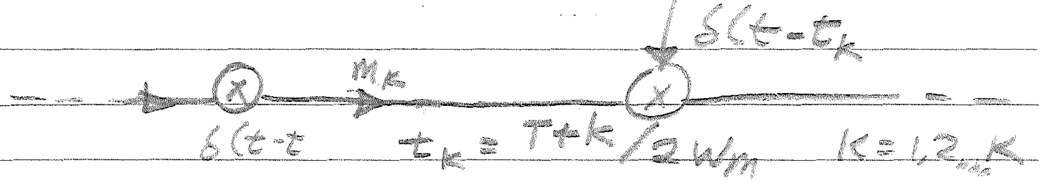
RECEIVER ESTIMATES $\hat{m}_K$, AND THEN CONSTRUCTS ESTIMATED WAVEFORM: $\hat{m}(t) = \sum_{K=1}^{K} \hat{m}_K \psi_K(t)$

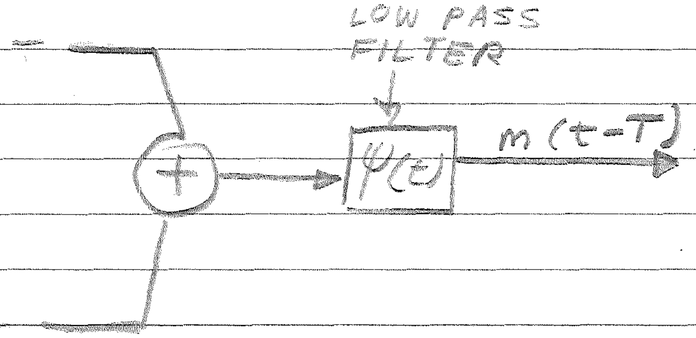
(NOTE $\hat{m}(t)$ IS BANDLIMITED)

MAXIMUM LIKLIHOOD

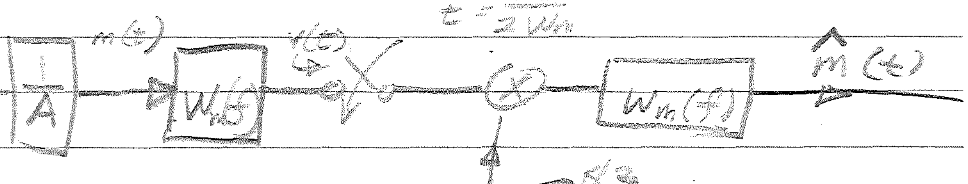








MAY ALSO REPLACE ϕ BY ψ, AND REPLACE BY LP FILTER



$-\frac{K}{2} \leq k \leq \frac{K}{2} \quad \sum_{-K/2}^{K/2} \frac{1}{2W_m}\delta\left(t-\frac{k}{2W_m}\right)$

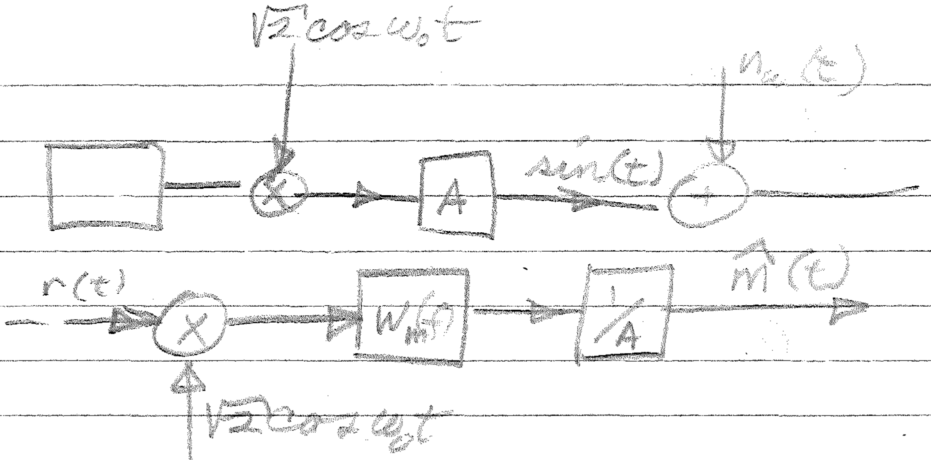
$r(t) = \int_{-\infty}^{\infty} m(\tau)\cdot\sqrt{2W_m}\,\varphi(t-\tau)d\tau$

$= m\sqrt{2W_m}\sum_{-K/2}^{K/2}\int_{-\infty}^{\infty}\varphi(\tau-k/2W_m)\,\psi(t-\tau)d\tau$

$\Rightarrow r\left(\frac{k}{2W_m}\right) = m\sqrt{2W_m}\sum_{-K/2}^{K/2}\varphi\left(\tau-\frac{k}{2W_m}\right)\psi\left(\frac{k}{2W_m}-\tau\right)d\tau$

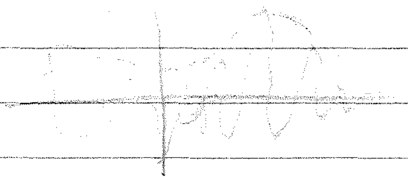
$\psi(t) = \varphi(-t)$

$\Rightarrow r\left(\frac{k}{2W_m}\right) = \sqrt{2W_m}\,m_k$



MAXIMUM LIKLIHOOD: DSB-SC.

DSB WITH CARRIER



READ 8-2   SKIP MATH

611-619   621-627

SKIP THRESHOLD   639-646

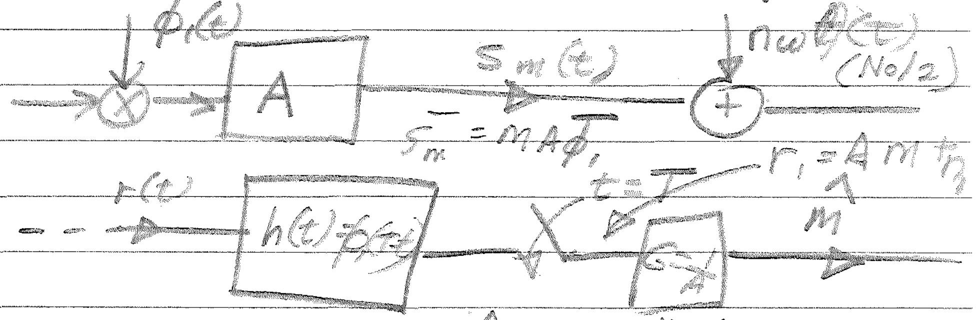
[illegible] 8-[illegible] BUT 656

MON

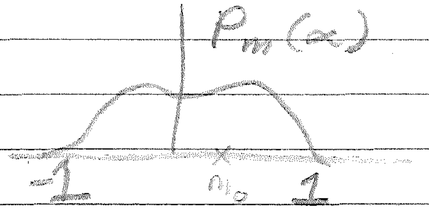
TWISTED MODULATION:

(NON LINEAR)

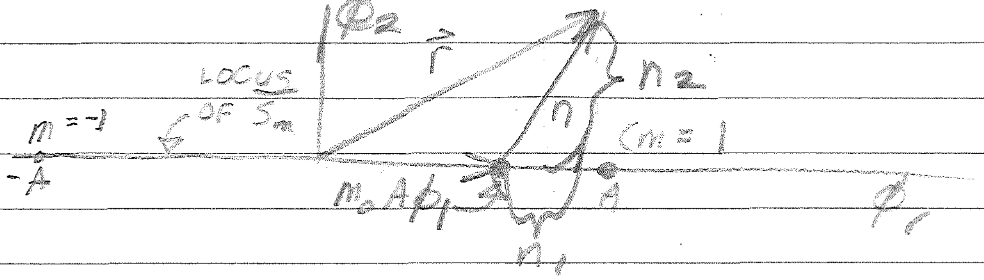
LINEAR MODULATION: $\bar{S}_m = m A \bar{\phi}_1$



$\hat{m} = m + n_1/A$



$$\overline{\varepsilon^2} = \overline{(\hat{m}-m)^2} = \left(\frac{\overline{n_1^2}}{A^2}\right) = \frac{N_0/2}{A^2}$$

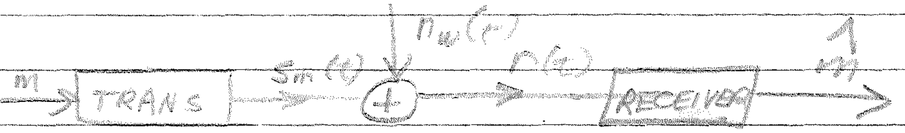


$A = \left|\frac{d\bar{S}_m}{dm}\right|$ (THE "STRETCH OF THE SIGNAL LOCUS

DEFINE STRETCH:

$S \triangleq \left|\frac{d\bar{S}_m}{dm}\right|$ $\quad \overline{\varepsilon^2} = \frac{N_0/2}{S^2}$ FOR LINEAR MOD.

$A = S$

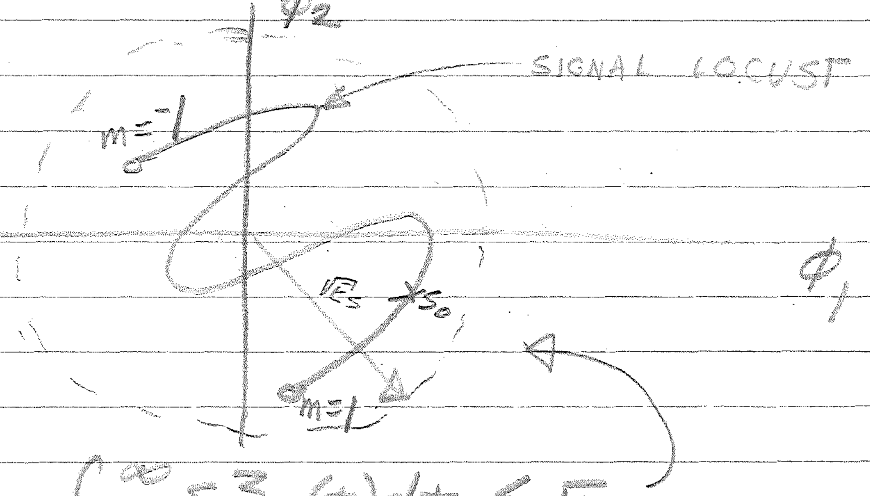


$$s_m(t) = a(m)\phi_1(t) + a_2(m)\phi_2(t)$$

$$\delta_{ij} = \int_{-\infty}^{\infty} \phi_1(t)\phi_j(t)dt$$

$$\Rightarrow s_m(t) = a_1(m)\overline{\phi}_1 + a_2(m)\overline{\phi}_2$$

FOR LINEAR MOD., $a_1(m)$ and $a_2(m)$ WOULD BE LINEAR FUNCTIONS OF M. FOR NON-LINEAR (TWISTED) COMM $a_1(m)$ & $a_2(m)$ ARE NON LINEAR FUNCTIONS OF M



IF $\int_{-\infty}^{\infty} s_m^2(t)dt \leq E_S$

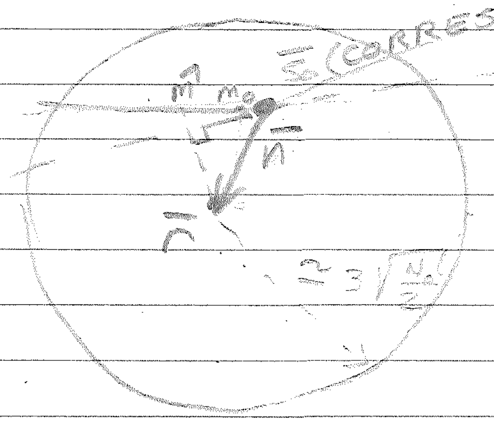
ASSUME:

$S = \left|\frac{ds_m}{dm}\right| = \frac{L}{2}$, A CONSTANT INDEPENT OF M. L=TOTAL LENGTH OF LOCUST.

$S = \frac{L}{2}$

WEAK NOISE SUPPRESION

$\overline{\epsilon^2} = \frac{N_0/2}{S^2}$



(CORRESPONDS TO TRAS INPUT No)

$\overline{S_m} \simeq \overline{S_0} + (m - m_0)\overline{S_0}$

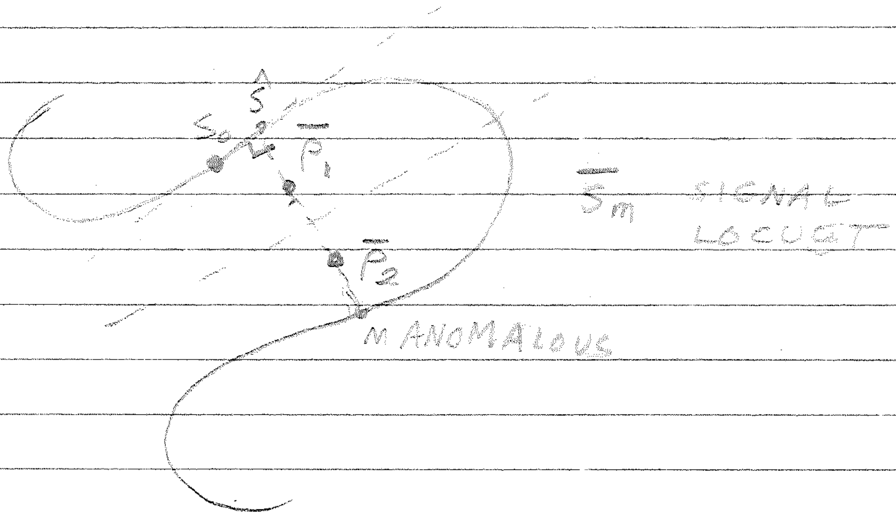
$\Rightarrow S_0 = dS_m/dm \,\Big|_{m = m_0 \text{ OR } S_m = S_0}$

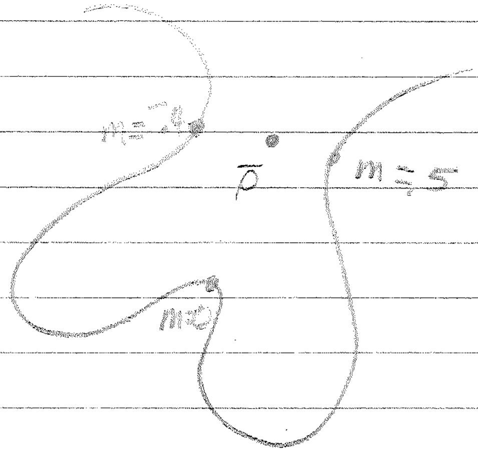
CONDITIONAL: $\overline{\epsilon^2}$

$E\left[(m-\hat{m})^2 / m = m_0\right] = \frac{N_0/2}{|\overline{S_0}|^2}$

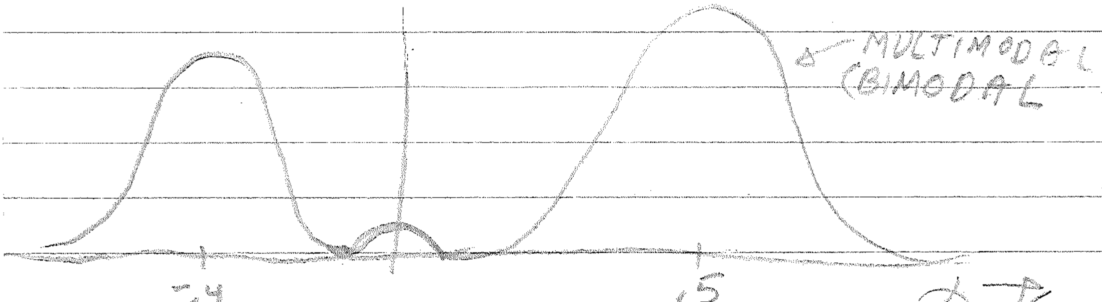
UNCONDITIONAL $\overline{\epsilon^2}$

$\overline{\epsilon^2} = \frac{N_0/2}{S^2} = \frac{N_0/2}{L^2/4}$





$P_m(\alpha|\bar{r}=\bar{\rho})$ APOSTEORIA



$$E[m|\bar{r}=\bar{\rho}]=\int_{-\infty}^{\infty}\alpha p_m(\alpha|r=\rho)\,d\alpha$$

MAXIMUM LIKIHOOD RECEIVER DESIGN

FOR ADDITIVE W.G.N.

$$p_{m,r}(\alpha,\rho) = p_r(\rho \mid m = m_0)\, p_m(\alpha)$$

$$= p(\rho)\, p_m(\alpha \mid r = \rho)$$

$$\Rightarrow p_m(\alpha \mid r = \rho) = \frac{p_r(\rho \mid m = \alpha)\, p_m(\alpha)}{p_r(\rho)}$$

$$p_r(\bar{\rho} \mid m = \alpha) \propto e^{-|\rho - \bar{s}|^2 / N_0}$$

MAXIMIZE $\rho \cdot s_\alpha - \frac{1}{2} E_\alpha$

WISH NO MAXIMIZE

$$(\bar{\rho} - s_\alpha)(\bar{\rho} - \bar{s}_\alpha) = \underset{\text{NOT } f(\alpha)}{|\bar{\rho}|^2} - \underset{E_\alpha}{|\bar{s}_\alpha|^2} + 2\bar{\rho} \cdot \bar{s}_\alpha$$

$$\rho \cdot s_\alpha - \frac{1}{2} E_\alpha = \int_{-\infty}^{\infty} \rho(t)\, s_\alpha(t)\, dt - \frac{1}{2} E_\alpha$$

CORRELATION DETECTION

LINEAR MODULATION:

$$s_\alpha(t) = \alpha A \phi(t)$$

THUS $\int_{-\infty}^{\infty} \rho(t)\, s_\alpha(t)\, dt - \frac{1}{2} E_\alpha$

$$= \alpha A \int_{-\infty}^{\infty} \rho(t)\, \phi(t) - \frac{1}{2} E_\alpha$$

$$= \alpha A \rho - \frac{1}{2} \alpha^2 A^2$$

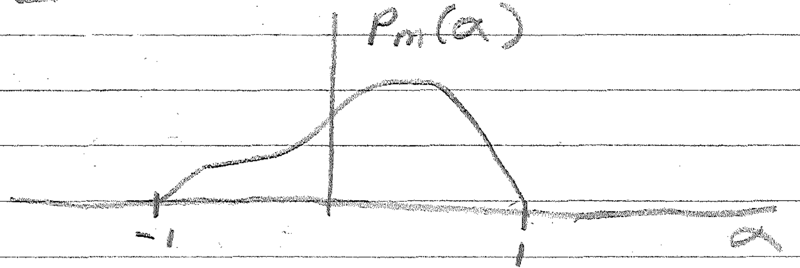
DIFFERENTIATE TO FIND MAX

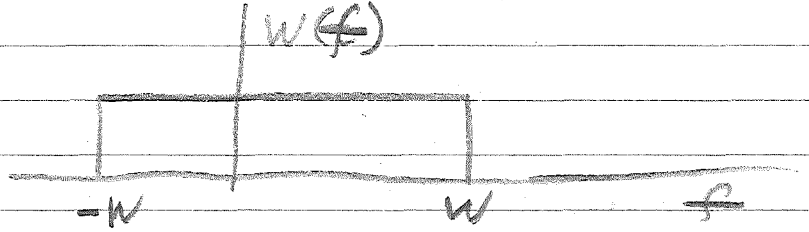
$$\Rightarrow \hat{\alpha} = \rho / A = \alpha + \frac{n}{A}$$

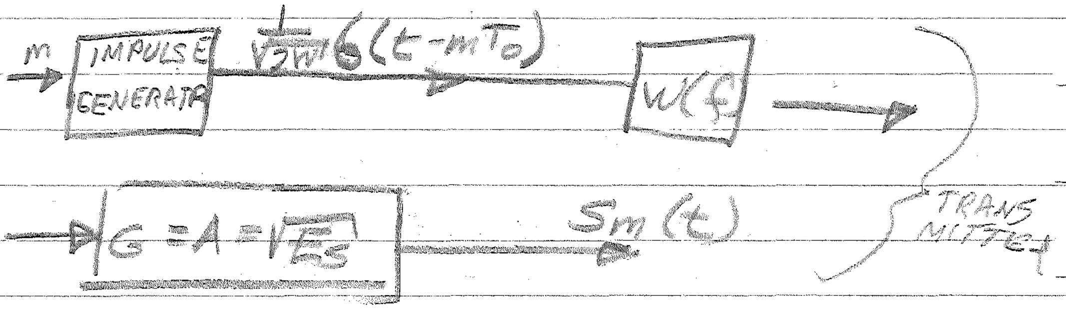
MON

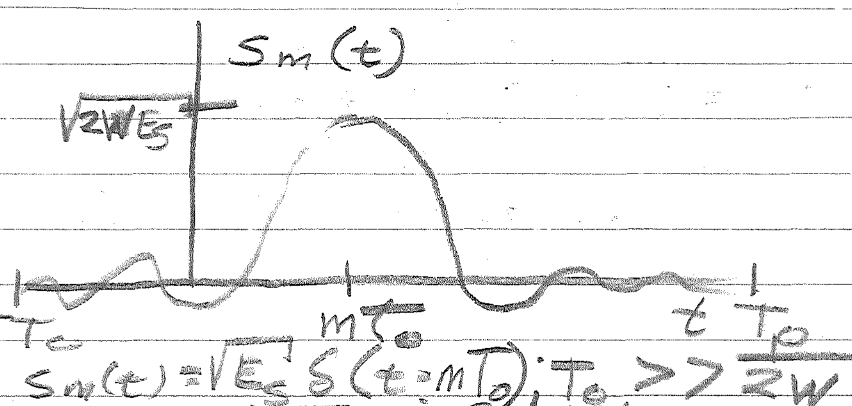
PULSE POSITION MODULATION

$$\left[\overline{\varepsilon^2} = \frac{N_0/2}{S^2} = \left(\frac{N_0/2}{2E_s}\right) \text{FOR LINEAR MODULATION}\right]$$









$s_m(t) = \sqrt{E_s}\,\delta(t - mT_0)$; $T_0 >> \frac{T_0}{2W}$

$\phi(t) = \sqrt{2W}\,\frac{\sin 2\pi W t}{2\pi W t}$ $\quad -W < t < W$

TRANSMITTED ENERGY $|\bar{S}_m|^2 = E_s$

MAXIMUM LIKELIHOOD RECEIVER:

$P_r(\bar{\rho} | m = \alpha)$ IS TO BE MAXED

$$\propto \int_{-\infty}^{\infty} \rho(\gamma) s_m(\gamma) d\gamma \qquad P_m(\alpha | r = \rho) = \frac{P_r(\rho | m = \alpha) P_m(\alpha)}{P_r(\rho)}$$

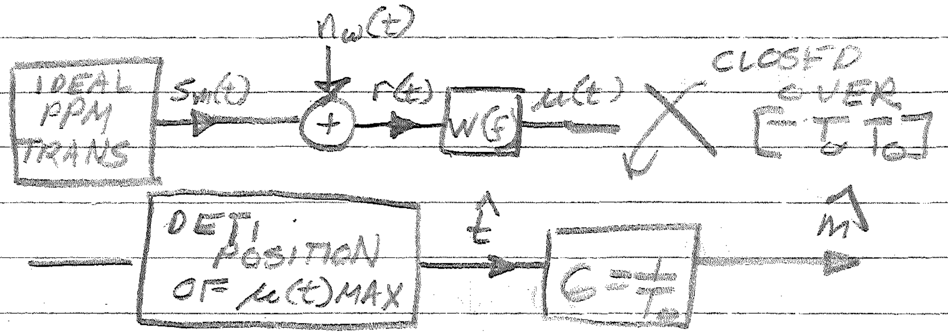
$$= \alpha \sqrt{E_s} \int_{-\infty}^{\infty} \rho(\gamma) \phi(\gamma - \alpha T_0) d\gamma$$

THIS IS THE OUTPUT OF A FILTER MATCHED TO $\phi(-t)$ i.e. $h(t) = \phi(-t)$ WHEN INPUT IS $\rho(t)$

$$\xrightarrow{\rho(t)} \boxed{h(t)} \longrightarrow$$

$$\text{OUTPUT} = \int_{-\infty}^{\infty} \rho(\tau) h(t - \tau) d\tau$$
$$= \int_{-\infty}^{\infty} \rho(\tau) \phi(\tau - t) d\tau$$
$$= \int_{-\infty}^{\infty} \rho(\tau) \phi(\tau - \alpha T_0) d\tau$$

SINCE $\phi(t) = \phi(-t)$ (FOR A LP FILTER), A MAXIMUM LIKELIHOOD RECEIVER PASSES $\rho(t)$ THRU A LP FILTER $W(f)$, THEN DETERMINES THE TIME INSTANT, $\hat{t}$, $-T_0 \le t \le T_0$ AT WHICH THE OUTPUT IS MAX, AND SETS $\hat{m} = \hat{t}/T_0$



8.79a) $S^2 = \int_{-\infty}^{\infty} \left[\frac{\delta}{\delta m} s_m(t)\right]^2 dt$

$s_m(t) = \sqrt{E_s}\,\phi\,(t - mT_0)$

$\frac{\delta s_m(t)}{\delta m} = -\sqrt{E_s}\,T_0\,\phi'(t - mT_0)$

$\Rightarrow S^2 = \;\; \sqrt{E_s}\,T_0 \int_{-\infty}^{\infty} \left[\phi'(t\,mT_0)\right]^2 dt$

$\phi(t) = \sqrt{2W}\;\; \frac{\sin 2\pi W t}{2\pi W t}$

USING PARSIVAL'S THEM:

$S^2 = E_s T_0{}^2 \int_{-W}^{W} |j\,2\pi f|^2 \frac{1}{2W}\,df$

$S^2 = \frac{E_s}{3}(2\pi T_0 W)^2$

NOTE: S^2 INDEPENDENT OF m

$\Rightarrow \overline{\varepsilon^2} = \frac{N_0/2}{S^2} = \frac{12}{\pi^2}\left(\frac{1}{4T_0W}\right)^2 \frac{N_0}{2E_s}$

FOR $\underset{\text{ULSE}}{P}\underset{\text{MP}}{A}M = \frac{N_0}{2E_s}$

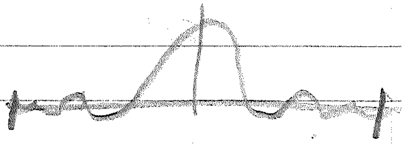
[FOR LIN MOD.

$(m_1 + m_2)\phi(t) = m_1\phi(t) + m_2\phi(t)$

$\overline{\varepsilon^2} = N_0/2E_s$]

CONSIDER $4T_0W$; SIGNAL BW = W

SIGNAL INTERVAL $T_s = 2T_0$



$\Rightarrow \underbrace{4T_0W}$ = TWICE BW, X SIGNAL INTERVAL

SAME # ~~EACH~~ OF SAMPLES TO

COMMUN. $-T_0 < s(t) < T_0$ AND $-W < S(f) < W$

CALLED EFFECTIVE DIMENSIONALITY

$\rightarrow 4T_0W = \beta$

$\therefore \overline{\varepsilon^2} = \frac{12}{\pi^2}\frac{1}{\beta^2}\left(\frac{N_0}{2E_s}\right)$

WAVEFORM COMM. WITH P.P.M.

LOW PASS WAVEFORM

$m(t) = \sum_{k=-\infty}^{\infty} m_k \phi_k(t);\ |f| \leq W_m$

ASSUME $|m_k| \leq 1 \ \forall k$

? W_m

WITH P.P.M., EACH MESSAGE, m_k IS TRANSMITTED AND ESTIMATED BY A MAXIMUM LIKELIHOOD RECEIVER IN SUCCESSION, AND

PRODUCES $\hat{m}_K$. THE RECEIVER CONSTRUCTS

$$\hat{m}(t) = \sum_{K=-\infty}^{\infty} \hat{m}_K \psi(t)$$

$$\hat{m}_K = m_K + n_K$$

$$\Rightarrow \hat{m}(t) = m(t) + n(t)$$

$$n(t) = \sum_{K=-\infty}^{\infty} n_K \psi_K(t)$$

= STATIONARY GAUSSIAN PROCESS WITH SPECTRAL DENSITY

$$S_n(f) = \begin{cases} \overline{\epsilon^2} & |f| \leq W_m \\ = 0 & \text{ELSEWHERE} \end{cases}$$

AVE. OUTPUT NOISE PWR

$$= \overline{n^2(t)} = 2W_m S_n(f)$$

$$= 2W_m \overline{\epsilon^2}$$

OR $\overline{n^2(t)} = \frac{12}{\pi^2} \frac{1}{\delta^2} \frac{N_0 W_m}{E_s}$

FOR LINEAR MOD:

$$\overline{n^2(t)} = N_0 W_m / E_s$$

TIME AVAILABLE FOR TRANSMISSION OF EACH m_k IS THE SAMPLING INTERLAL $\frac{1}{2W_m}$. THUS, MAX ALLOWABLE POSITION DEVIATION FOR EACH TRANSMISSION IS $2T_0 \le \frac{1}{2W_m}$. IF WE SET $2T_0 = 1/2W_m$, THEN WE GET

$$4T_0W = B = W/W_m = \text{B.W EXPANSION RATIO}$$

FREQUENCY POSITION MODULATION

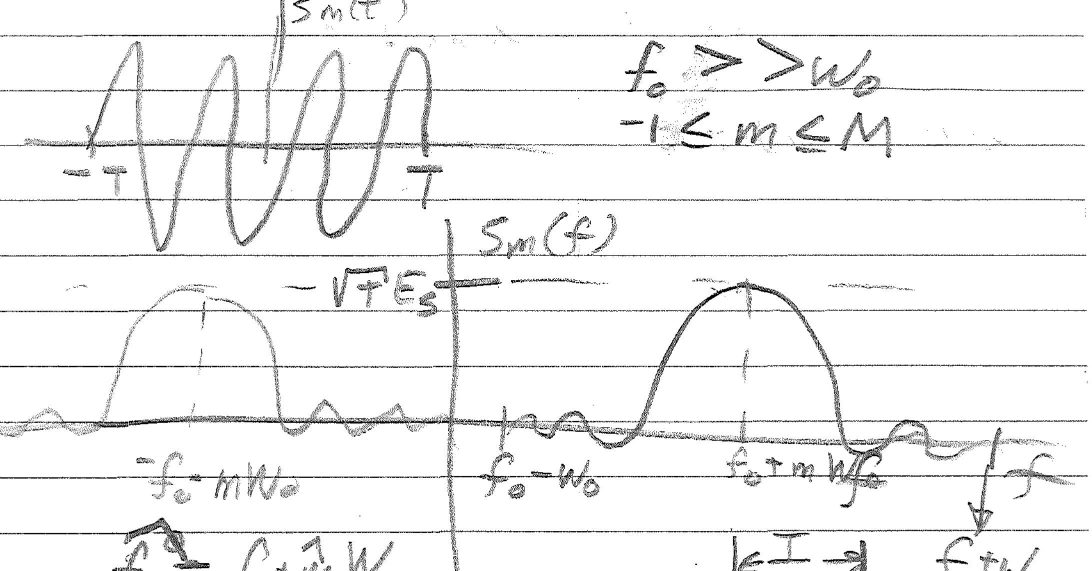
$$S_m(t) = \sqrt{E_S}\,\phi_m(t)$$

$$\phi_m(t) = \frac{1}{\sqrt{T}} \cos 2\pi(f_0 + W_0 t); \; -T<t<$$

$$= 0 \quad \text{OTHERWISE}$$

$f_0 >> W_0$

$-1 \le m \le M$



$$\hat{f} = f_0 + \hat{m} W_0$$

$$\hat{m} = \frac{\hat{f} - f_0}{W_0}$$

$$B = 4TW_0$$

$$S^2 = \frac{E_S}{3}(2\pi T W_0)^2$$

$$\overline{n^2(t)} = \frac{12}{\pi^2}\left(\frac{W_m}{W_0}\right)^2 \frac{N_0 W_m}{E_s}$$

MON

FM

$$S_m(t) = A\sqrt{2}\cos 2\pi\left[\underbrace{f_0 t + W_1 \int m(t)\,dt}_{\text{INSTANTANEOUS PHASE}}\right]$$

$$f_{INST} = \frac{d}{dt}\left[f_0 t + W \int m(t)\,dt\right]$$
$$= f_0 + W_1\, m(t)$$

SPECTRUM: LET $m(t) = \cos 2\pi W_m t)$

$$w_0 = 2\pi f_0 \quad w_m = 2\pi W_m$$

$$S_m(t) = A\sqrt{2}\cos\left(w_0 t + \frac{W_1}{W_m}\sin w_m t\right)$$

$$\beta = W_1/W_M$$

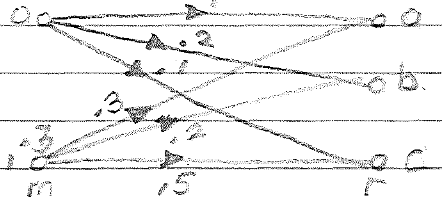
$$S_m(t) = A\sqrt{2}\left[\cos(\beta \sin wt)\cos w_0 t - \sin(\beta \sin w_m t)\sin w_0 t\right]$$

$$J_K(\beta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i}$$

BOB
MARKS

30/30

2.12)



a) $P[0,a] = .49$ $P[1,a] = .09$

$P[0,b] = .14$ $P[1,b] = .06$

$P[0,c] = .07$ $P[1,c] = .15$

$\hat{m}(a) = 0$ (.49)

$\hat{m}(b) = 0$ (.14)

$\hat{m}(c) = 1$ (.15)

$P[C] = .78 \Rightarrow P[E] = .22$

b) 1) $P[0,0,0] = C = P(0)$ $P(\mathcal{E}_0) = 1 - P[0]$

2) $P[0,0,1] = C = (.7)P[0] + (.2)P[0] + (.5)P[1]$

$= (.9)\ P[0] + (.5)[1 - P[0])$

$= .4\ P[0] + .5$

$P[E] = 1 - P[C] = .5 - .4P[0]$

3) $P[0,1,0] \Rightarrow P[C] = (.8)P[0] + (.2)[1 - P[0]] = .2 + .6P[0]$

$P(\mathcal{E}) = 1 - P[C] = .8 - .6P[0]$

4) $P[0,1,1] = (.7)P[0] + (.7)[1 - P[0]] = .7 \Rightarrow P[\mathcal{E}] = .3$

5) $P[1,0,0] = (.3)[1 - P[0]] + .3P[0] = .3 \Rightarrow P[\mathcal{E}] = .7$

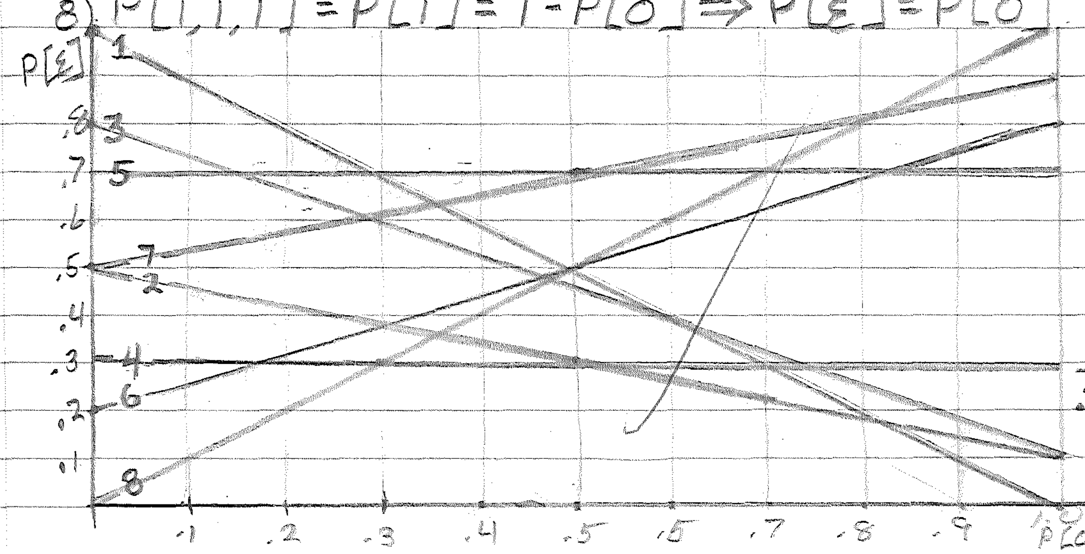
6) $P[1,0,1] = (.8)[1 - P[0]] + .2P[0] = .8 - .6P[0]$

$\Rightarrow P[\mathcal{E}] = .2 + .6P[0]$

7) $P[1,1,0] = (.5)(1 - P[0]) + .1P[0] = .5 - .4P[0]$

$\Rightarrow P[\mathcal{E}] = .5 + .4P[0]$

8) $P[1,1,1] = P[1] = 1 - P[0] \Rightarrow P[\mathcal{E}] = P[0]$



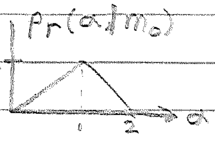
	MAX
1	1.0
2	.5
3	.8
4	.3
5	.7
6	.8
7	.9
8	1.0

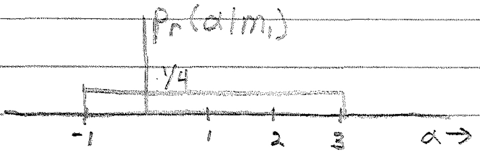
$\therefore\ (\overset{a}{0}, \overset{b}{1}, \overset{c}{1})$

IS MINIMAX

BOB MARKS

2.23) a)



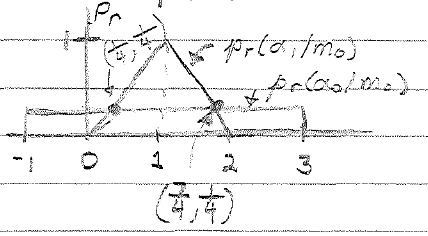


$p[m_0] = p[m_1]$

WHEN $\hat{m}(\alpha) = m_0$

IFF $p_r(\alpha/m_0)P[m_0] > p_r(\alpha/m_1)P[m_1]$

OR $p_r(\alpha|m_0) > p_r(\alpha/m_1)$



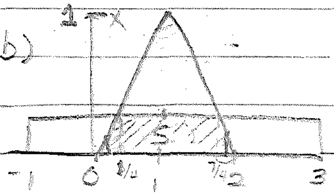
$p_r(\alpha/m_0) > p_r(\alpha/m_1)$ FOR $\frac{1}{4} < \alpha < \frac{7}{4}$

$p_r(\alpha/m_1) > p_r(\alpha/M_0)$ FOR $-1 < \alpha < \frac{3}{4}, \frac{7}{4} < \alpha < 3$

$\therefore \hat{m}(\alpha) = m_0 \quad \frac{1}{4} < \alpha \frac{7}{4}$

$\hat{m}(\alpha) = m_1 \quad -1 < \alpha \frac{3}{4}, \frac{7}{4} \alpha < 3$

b)



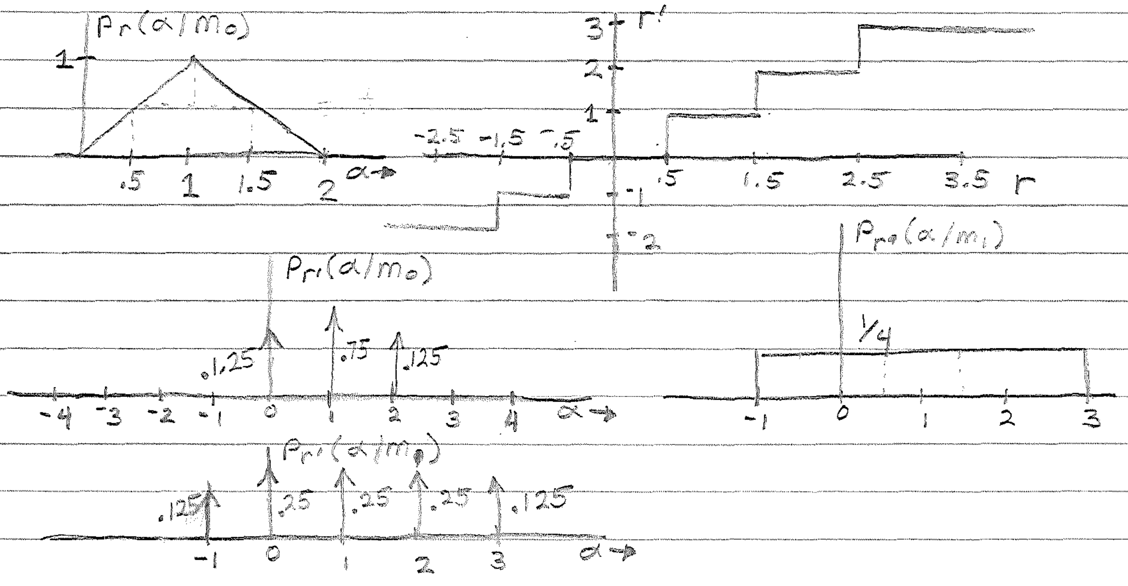
$\int x\,d\alpha = 1 + \left(\frac{3}{4}\right)^2 = 1\frac{9}{16} = \frac{32}{16}$

$S = \left(\frac{3}{2}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)^2 = \frac{7}{16}$

$\Rightarrow P(E) = 7/32 = .22$

BOB
MARKS

2.24)



$P(m_0) = P(m_1) \Rightarrow P(m_0) = P(m_1) = .5$

$P(-2, m_0) = 0$ $\quad$ $P(-2, m_1) = 0$

$P(-1, m_0) = 0$ $\quad$ $P(-1, m_1) = (.125)(.5) = .0625$

$P(0, m_0) = (.125)(.5) = .0625$ $\quad$ $P(0, m_1) = (.25)(.5) = .125$

$P(1, m_0) = (.75)(.5) = .375$ $\quad$ $P(1, m_1) = (.25)(.5) = .125$

$P(2, m_0) = (.125)(.5) = (.0625)$ $\quad$ $P(2, m_1) = (.25)(.5) = .125$

$P(3, m_0) = 0$ $\quad$ $P(3, m_1) = (.125)(.5) = .0625$

$\hat{m}(-2) = m_0$ OR m_1

$\hat{m}(-1) = m_1$ $\quad$ .0625

$\hat{m}(0) = m_1$ $\quad$ .1250

$\hat{m}(1) = m_0$ $\quad$ .3750

$\hat{m}(2) = m_1$ $\quad$ .1250

$\hat{m}(3) = m_1$ $\quad$ $\underline{.0625}$

$.7500 \Rightarrow P[E] = .25$

BOB MARKS

30/40

3-4) $R_i(\tau) = \overline{n_i(t)\, n_i(t-\tau)} = \frac{\sin \pi\tau}{\pi\tau}$; $i = 1, 2$

$R_{12}(\tau) = \overline{n_1(t)\, n_2(t-\tau)} = \frac{\sin \pi\tau}{2\pi\tau}$

$x_1 = n_1(t)\big|_{t=0}$

$x_2 = n_1(t)\big|_{t=1}$

$x_3 = n_2(t)\big|_{t=0}$

$(x_1, x_2):\ R_1(\tau)\big|_{\tau=1} = 0$

$(x_1, x_3):\ R_{12}(\tau)\big|_{\tau=0} = 1/2$

$(x_2, x_3):\ R_{12}(\tau)\big|_{\tau=1} = 0$

$\Rightarrow x_2$ INDEPENDENT OF x_1 AND x_3

$\rho_{13} = 1/2$; $\sigma_1 = \sigma_2 = 1$

$$\Rightarrow p_{x_1 x_3}(\alpha, \beta) = \frac{1}{2\pi\sqrt{3/4}} \exp\left[-\frac{\alpha^2 - \alpha\beta + \beta^2}{2\,(3/4)}\right]$$

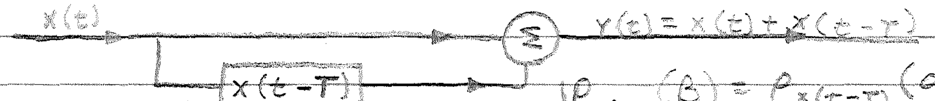
$$p_{x_2}(\gamma) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{\gamma^2}{2}\right]$$

$$p_{x_1 x_2 x_3}(\alpha, \gamma, \beta) = \left(p_{x_1 x_3}(\alpha, \beta)\, p_{x_2}(\gamma)\right)$$

$$= \left(\frac{1}{\sqrt{6}\,\pi^{3/2}} \exp\left[-\frac{\alpha^2 - \alpha\beta + \beta^2}{(1.5)} - \frac{\gamma^2}{2}\right]\right)$$

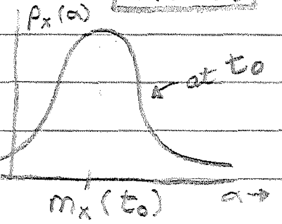
densities do not depend on time.

3-6)

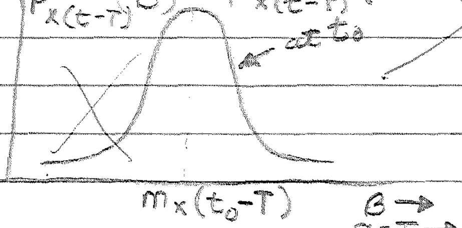


$Y(t) = x(t) + x(t-T)$

a)



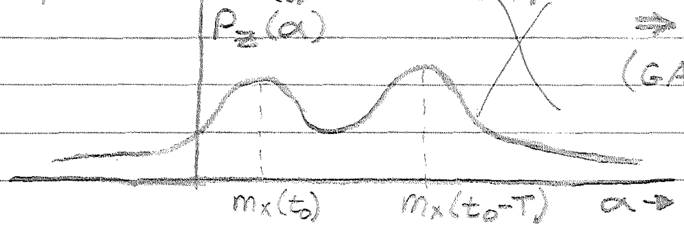
$P_{x(t-T)}(\beta) = P_{x(t-T)}(\alpha - T)$



$\beta = \alpha - T$

Sum of two Gaussian r.v. is Gaussian.

$P_Y(\alpha) = \frac{1}{2}\left(P_{x(t_0)}(\alpha) + P_{x(t_0-T)}(\alpha - T)\right)$



$\Rightarrow p_Y(\alpha)$ NOT GAUSSIAN (GAUSSIAN IFF $m_x(t) = m_x$ OR $T = 0$)

b) $\mathcal{L}_x(t,s) = E\left[(x(t) - m_x(t))(x(s) - m_x(s))\right]$

$m_Y(t) = \int_{-\infty}^{\infty} m(\gamma)\, h(t-\gamma)\, d\gamma + m_x(t)$

$h(t) = \delta(t-T)$

$\Rightarrow m_Y(t) = m_x(t) + m_x(t-T)$

$\mathcal{L}_Y(t,s) = E\left[(Y(t) - m_Y(t))(Y(s) - m_Y(s))\right]$

$Y(t) = x(t) + x(t-T)$ — express in terms of $\mathcal{L}x(a,b)$

$\Rightarrow \mathcal{L}_Y(t,s) = E\left[\{x(t) + x(t-T) - m_x(t) - m_x(t-T)\}\{x(s) + x(s-T) - m_x(s) - m_x(s-T)\}\right]$

c) STATIONARITY $\Rightarrow m_x(t) = m_x$; $\overline{x(t)} = \overline{x(t+T)}$

$\Rightarrow \mathcal{L}_Y(t,s) = E\left[(x(t) + x(t-T) - 2m_x)(x(s) + x(s-T) - 2m_x)\right]$

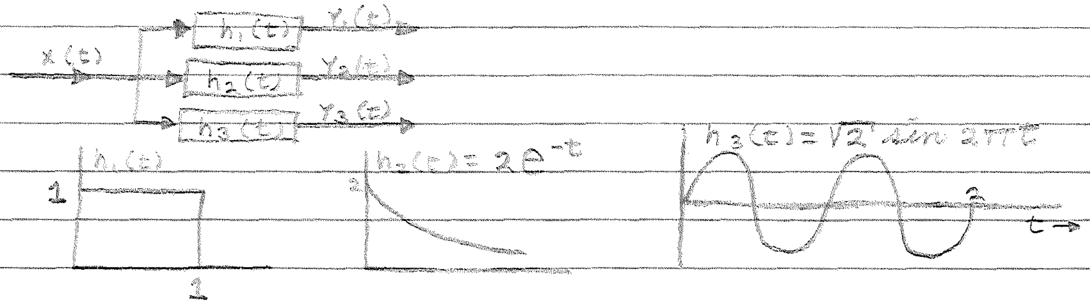
$\underline{R_Y(t,s)} = E\left[(x(t) + x(t-T))(x(s) + x(s-T))\right]$

1) $m_Y(t) = 2m_x =$ CONSTANT

2) $\mathcal{L}_Y(t-s, 0) = E\left[(x(t-s) + x(t-s-T) - 2m_x)(x(0) + x(-T) - 2m_x)\right]$

$= \mathcal{L}_Y(t,s) \Rightarrow$ STATIONARITY

3-7)



a) $\overline{Y_i(t)} = \overline{X(t)} \int_0^\infty h(\lambda)d\lambda$

$\overline{X(t)} = 0 \Rightarrow \overline{Y_i(t)} = 0 \; ; \; i = 1, 2, 3$

$\overline{Y_i^2(t)} = \frac{N_0}{2} \int_0^\infty h^2(\lambda)d\lambda$

$$\begin{cases} \overline{Y_1^2(t)} = \frac{N_0}{2} \int_0^1 d\lambda = \frac{N_0}{2} \\ \overline{Y_2^2(t)} = 2N_0 \int_0^\infty e^{-2\lambda} d\lambda = -N_0 e^{-2\lambda}\Big]_0^\infty = N_0 \\ \overline{Y_3^2(t)} = N_0 \int_0^2 \sin^2 2\pi\lambda d\lambda = N_0 \left[\frac{\lambda}{2} - \frac{\sin 4\pi\lambda}{8\pi}\right]_0^2 = N_0 \end{cases}$$

b) $\overline{Y_i(t)Y_j(t)} = \frac{N_0}{2} \int_0^\infty h_i(\lambda)h_2(\lambda)d\lambda = \overline{Y_j(t)Y_i(t)}$

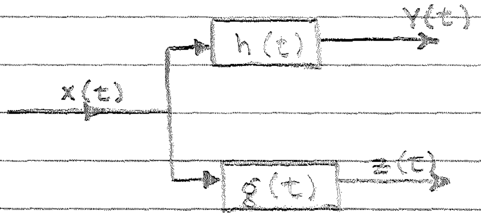
$$\begin{cases} \overline{Y_1(t)Y_2(t)} = N_0 \int_0^\infty (u(t) - u(t-1))e^{-t}dt \\ \qquad = N_0 \int_0^1 e^{-t}dt = -N_0 e^{-t}\Big]_0^1 = N_0[1 - e^{-1}] \\ \overline{Y_1(t)Y_3(t)} = \frac{N_0}{\sqrt{2}} \int_0^1 \sin 2\pi t dt = 0 \\ \overline{Y_2(t)Y_3(t)} = \sqrt{2}\, N_0 \int_0^2 e^{-t} \sin 2\pi t dt \\ \qquad = \frac{-\sqrt{2}N_0}{1+4\pi^2}\left[e^{-t}(\sin 2\pi t + 2\pi \cos 2\pi t)\right]_0^2 \\ \qquad = \frac{+\sqrt{2}N_0}{1+4\pi^2}[1 - e^{-2}] \end{cases}$$

c) $\overline{Y_i(t)Y_j(s)} = \frac{N_0}{2} \int_0^\infty d\lambda_i \int_0^\infty h(\lambda_i) h(\lambda_j) d\lambda_j = \overline{Y_j(t)Y_i(s)}$

$$\begin{cases} \overline{Y_1(t)Y_2(s)} = N_0 \int_0^\infty d\lambda_1 \int_0^\infty h(\lambda_1) e^{-t} dt \\ \qquad = N_0 \int_0^\infty h(\lambda_1) d\lambda_1 = N_0 \\ \overline{Y_1(t)Y_3(s)} = \frac{N_0}{\sqrt{2}} \int_0^\infty d\lambda_2 \int_0^1 h(\lambda_2) d\lambda_1 \\ \qquad = \frac{N_0}{\sqrt{2}} \int_0^2 \sin 2\pi t dt = 0 \\ \overline{Y_2(t)Y_3(s)} = \sqrt{2} \int_0^\infty d\lambda_1 \int_0^2 \sin 2\pi t) h(\lambda_1) dt \end{cases}$$

$= 0$

3-8)



a) $R_x(\tau) = \mathcal{L}_x(\tau) + m_x^2$;

$\overline{Y} = m_Y = m_x \int_0^\infty h(\lambda)\,d\lambda$; $\overline{Z} = m_Z = m_x \int_0^\infty g(\lambda)\,d\lambda$

$\overline{Y^2} = \int_0^\infty R_x(\tau) h^2(\tau)\,d\tau$; $\overline{Z^2} = \int_0^\infty R_x(\tau)\, g^2(\tau)\,d\tau$

$\sigma_Y = (\overline{Y^2} - \overline{Y}^2)^{1/2}$; $\sigma_Z = (\overline{Z^2} - \overline{Z}^2)^{1/2}$

$\overline{YZ} = \int_0^\infty d\lambda_1 \int_0^\infty h(\lambda_1) g(\lambda_2)\,d\lambda_2$; $\rho_{XY} = (\overline{YZ} - \overline{Y}\,\overline{Z})/\sigma_Y \sigma_Z$

USING ABOVE VALUES

$$\rho_{Y,Z}(\alpha,\beta) = \frac{1}{2\pi\sigma_Y\sigma_Z\sqrt{1-\rho_{XY}}}\exp\left[\frac{-1}{2(1-\rho_{XY})}\left[\frac{(\alpha-\overline{Y})^2}{\sigma_Y^2} - \frac{2\rho_{YZ}}{\sigma_Y\sigma_Z} \times (\alpha-\overline{Y})(\beta-\overline{Z}) + \frac{(\beta-\overline{Z})^2}{\sigma_Z^2}\right]\right.$$

b) $\overline{Y(t)\,x(t-\tau)} = E\left[x(t-T)\,Y(t)\right]$

$= E\left[x(t-T)\int_0^\infty x(t-\lambda)h(\lambda)\,d\lambda\right]$

$= \int_0^\infty E\left[x(t-\tau)\,x(t-\lambda)\right]h(\lambda)\,d\lambda$

$= \int_0^\infty R_x(\tau-\lambda)h(\lambda)\,d\lambda$

$= \int_0^\infty \left[\mathcal{L}_x(\tau-\lambda) + m_x^2\right]h(\lambda)\,d\lambda$

$R_x(\tau) = S_o\,\delta(t)$

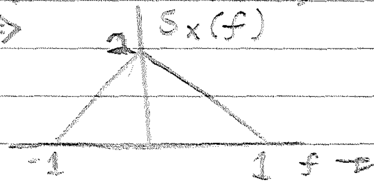
$\Rightarrow \overline{Y(t)x(t-\tau)} = \int_0^\infty S_o\,\delta(\tau-\lambda)h(\lambda)\,d\lambda = S_o\,h(\tau)$ ✓

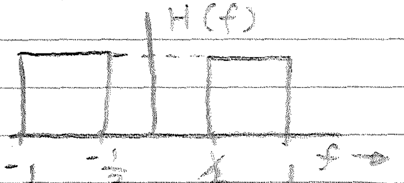
c) $\overline{Y(t)\,Z(t)} = \overline{Y(t)}\;\overline{Z(t)}$; $(\mathcal{L}(t_1,t_2) = 0)$

$\int_0^\infty d\lambda_1 \int_0^\infty h(\lambda_1) g(\lambda_2)\,d\lambda_2 = \int_0^\infty h(\lambda_2)\,d\lambda_2 \int_0^\infty g(\lambda_1)\,d\lambda_1$

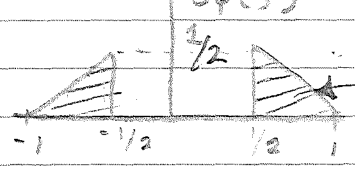
d) $m_x = 0 \Rightarrow \mathcal{L}_x(\tau) = R_x(\tau) = (\sin \pi\tau / \pi\tau)^2$

$S_x(f) = \mathcal{F}\{R_x(\tau)\} \Rightarrow$





$S_Y(f) = S_x(f)\,|H(f)|^2$



$S_Y(f) = 2 - |f| = \frac{dP}{df}$ FOR $\frac{1}{2} < |f| < 1$

$\Delta P = \int_{f_1}^{f_2} (2 - |f|)\,df$ $\qquad \frac{1}{2} < f < 1$

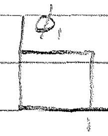
0 ELSEWHERE

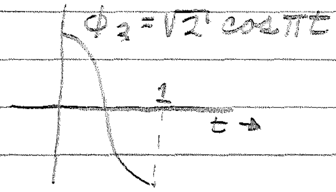
$\overline{Y^2} = \int_{-\infty}^{\infty} S_Y(f)\,df = \frac{1}{4}$

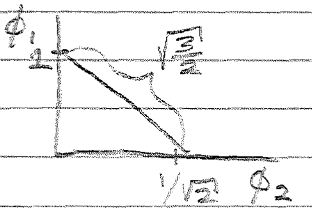
BOB MARKS

$\frac{46}{50}$

4-8) a)



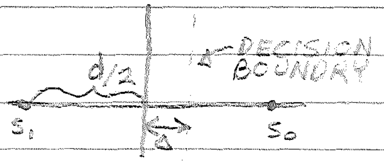




$\frac{N_o}{2} = .15$

$$\cancel{P[\mathcal{E}] = Q\left[\frac{d}{\sqrt{2N_o}}\right] = Q\left[\sqrt{\frac{3}{4(.3)}}\right]}$$
$$\cancel{= Q[1.5] = .06}$$

FOR $\left(\frac{1}{4}, \frac{3}{4}\right)$



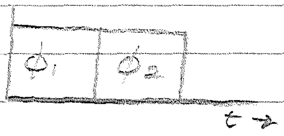
$$\Delta = \frac{N_o/2}{d} \ln \frac{P[m_1]}{P[m_0]}$$

.41

$$= \frac{.3}{\sqrt{3/2}} \ln 3 = .405 \qquad d = 1.23 \qquad \sqrt{2N_o} = .775$$

$$P[\mathcal{E}] = P[m_0]\, Q\left[\frac{d-2\Delta}{\sqrt{2N_o}}\right] + P[m_1]\, Q\left[\frac{d+2\Delta}{\sqrt{2N_o}}\right]$$
$$= \frac{1}{4} Q[.413] + \frac{3}{4} Q[2.76]$$
$$= \frac{1}{4}(3 \times 10^{-1}) + \frac{3}{4}(2 \times 10^{-3}) = .075 + .0015 = \textcircled{.077}$$

b)



$s_0 = (2, 0)$

$s_1 = (2, 2) \Rightarrow d = 2$

$$P[\mathcal{E}] = Q\left[\frac{d}{\sqrt{2N_o}}\right] = Q[2.6] = \cancel{4} \times 10^{-3}$$

FOR $\left(\frac{1}{4}, \frac{3}{4}\right)$

$$2\Delta = \frac{N_o}{d} \ln 3 = (.15)(1.1) = .165$$

$$P[\mathcal{E}] = \frac{1}{4} Q\left[\frac{.35}{.775}\right] + \frac{3}{4} Q\left[\frac{3.65}{.775}\right]$$
$$= \frac{1}{4} Q[.45] + \frac{3}{4} Q[4.7]$$
$$= \frac{1}{4}(.3) + \frac{3}{4}(1.5 \times 10^{-7}) = \boxed{.075}$$

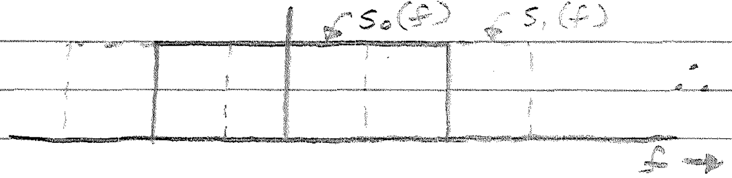
③

Should be lower when the messages are not equiprobable.

c) BY PARSIVAL'S THEM:

$$d^2 = \int_{-\infty}^{\infty} [s_1(f) - s_0(f)]^2 df$$

$$= \int_{-\infty}^{\infty} [s_1^2(f) - 2s_1(f)s_0(f) + s_0^2(f)] df$$



$\therefore\ d^2 = 2$

$$P[\varepsilon] = Q\left[\frac{d}{\sqrt{2N_0}}\right] = Q[1.83] = 4 \times 10^{-2}$$

FOR $\left(\frac{1}{4}\ \ \frac{3}{4}\right)$

$$\Delta = \frac{.15}{\sqrt{2}} \ln 3 = .117 \Rightarrow 2\Delta = .234 \qquad d = 1.414$$

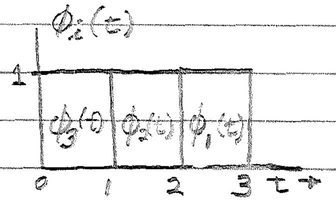
$$P[\varepsilon] = \frac{1}{4} Q\left[\frac{1.18}{.775}\right] + \frac{3}{4} Q\left[\frac{1.65}{.775}\right]$$

$$= \frac{1}{4} Q[1.5] + \frac{3}{4} Q[2.1]$$

$$= \frac{1}{4}(6 \times 10^{-2}) + \frac{3}{4}(10^{-2})$$

$$= 2.25 \times 10^{-2}$$

4-9)

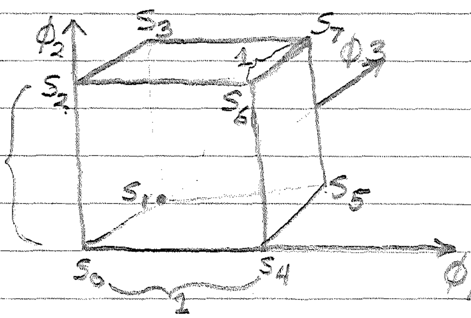


⇒ ORTHONORMAL BASIS FUNCTIONS

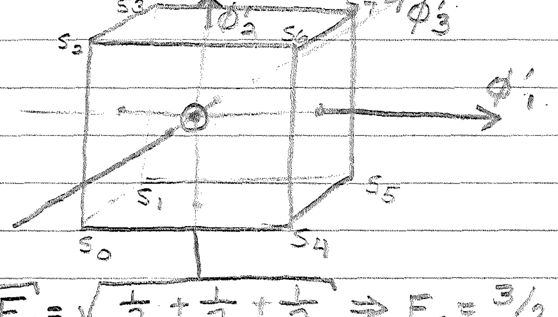
$\int_{-\infty}^{\infty} \phi_i(t)\phi_j(t)dt = 0 \qquad i \neq j$

$\int_{-\infty}^{\infty} \phi_i^2(t)dt = 1 \qquad i = 1,2,3$

$s_0 = (0,0,0) \qquad s_4 = (1,0,0)$
$s_1 = (0,0,1) \qquad s_5 = (1,0,1)$
$s_2 = (0,1,0) \qquad s_6 = (1,1,0)$
$s_3 = (0,1,1) \qquad s_7 = (1,1,1)$



TRANSLATION OF COORDINATES DOES NOT EFFECT VALUE OF P[C]. DEFINE $\phi_i'(t), i = 1,2,3$ PERPENDICULAR TO CENTER OF EACH SQUARE FACE:



$P[C] = (1-p)^N$

$p = Q\left(\sqrt{\frac{2E_s}{N N_0}}\right)$

$\sqrt{E_s} = \sqrt{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}} \Rightarrow E_s = 3/2$

$N = 3$

$\Rightarrow p = Q\left(\sqrt{\frac{3}{3N_0}}\right) = Q\left(\sqrt{\frac{1}{N_0}}\right)$

$\Rightarrow P[C] = (1 - Q(\sqrt{\frac{1}{N_0}})^3$

①

$\therefore P[E] = 1 - P[C] = 1 - [1 - Q(\sqrt{\frac{1}{N_0}})]^3$

✓

4-11) $s_0(t) = \sqrt{10^3 E}\, \cos 2\pi 10^6 t \qquad 0 < t < 2 \cdot 10^{-3}$

$s_1(t) = \sqrt{10^3 E}\, \cos 2\pi (10^6 + \Delta) t \qquad 0 < t < 2 \cdot 10^{-3}$

$$\frac{1}{10^3 E} \int_{-\infty}^{\infty} s_0(t) s_1(t)\, dt = \int_0^{10^{-3}} \cos 2\pi 10^6 t \cos 2\pi (10^6 + \Delta) t\, dt$$

$$= \frac{1}{2} \int_0^{2 \cdot 10^{-3}} \left(\cos -2\pi \Delta t + \cos 2\pi (2 \cdot 10^6 + \Delta) t \right) dt$$

$$= \frac{1}{2} \left[\frac{-1}{2\pi \Delta} \sin 2\pi \Delta t + \frac{1}{2\pi (2 \cdot 10^6 + \Delta)} \sin 2\pi (2 \cdot 10^6 + \Delta) t \right]_0^{2 \cdot 10^{-3}}$$

$= 0$ FOR $\Delta = 500, 250$

$\Rightarrow$ $S_0(t)$ AND $S_1(t)$ ARE ORTHOGONAL

$10 \log_{10} E/s = 10 \log_{10} 6$

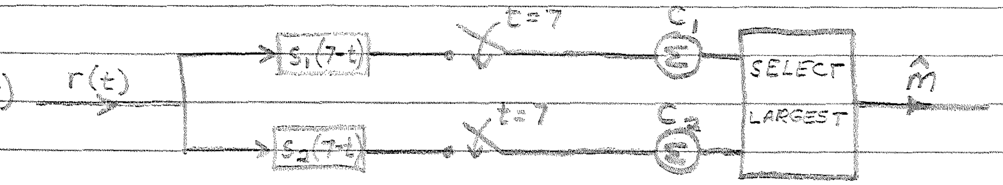
$= 7.78$

FROM FIG. 4-37:

$P[E] \doteq .0093$

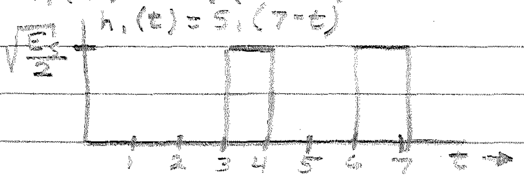
.0071

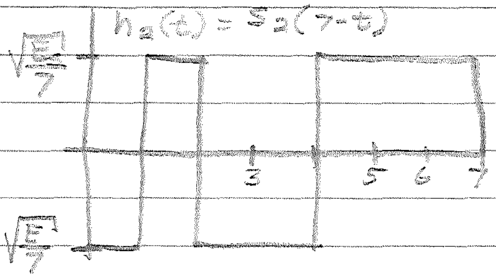
4-14) a)



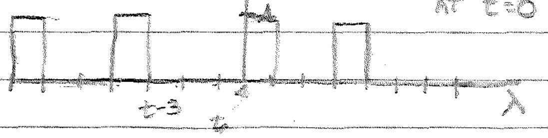
$$c_i = \frac{1}{2}\left(N_0 \ln P[m_i] - |s_i|^2\right)$$

$h_1(t) = s_1(7-t)$

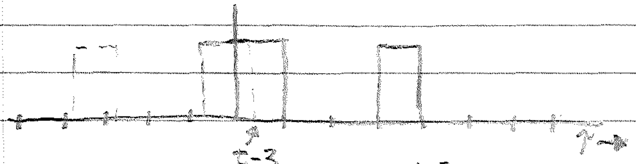




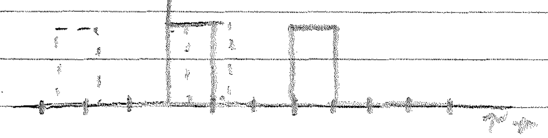
GIVEN $s_1(t)$ INTO FILTER WITH IMPULSE RESPONSE $h_1(t)$, OUTPUT $U_1(t) = s_1(t) * h_1(t)$
AT t=0 $(A^2 = E/7)$



$s_1(t) * h_1(t) = 0 \quad -\infty \leq t \leq 3$

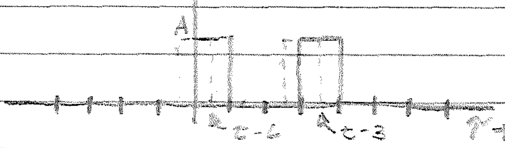


$s_1(t) * h_1(t) = \int_0^{t-3} A^2 d\tau = A^2(t-3) \quad 3 \leq t \leq 4$



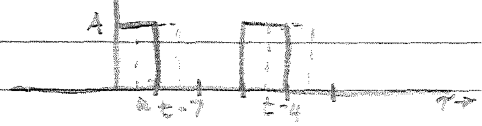
$s_1(t) * h_1(t) = \int_{t-1}^{1} A^2 dt = A^2[t]_{t-4}^{1} = A^2[1-t+4] = A^2(5-t)$
$4 \leq t \leq 5$

$s_1(t) * h_1(t) = 0 \quad 5 \leq t \leq 6$



$$s_1(t)*h_1(t)=\int_3^{t-3}A^2d\tau+\int_0^{t-6}A^2d\tau$$

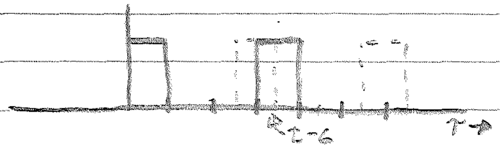
$$=A^2[t-3-3]+A^2[t-6]=A^2[2t-12] \qquad 6\le t\le 7$$



$$s_1(t)*h_1(t)=\int_{t-7}^{1}A^2d\tau+\int_{t-4}^{4}A^2d\tau$$

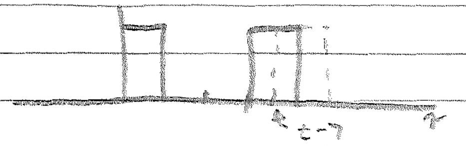
$$=A^2[1-t+7+4-t+4]=A^2[-2t+16] \qquad 7\le t\le 8$$

$s_1(t)*h_1(t)=0 \quad ; \quad 8\le t\le 9$



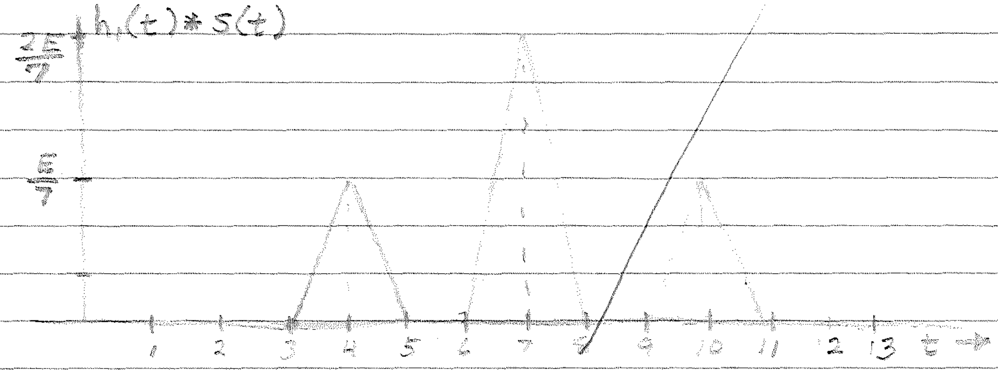
$$s_1(t)*h_1(t)=\int_3^{t-6}A^2d\tau=A^2[t-6-3]=A^2[t-9]$$

$9\le t\le 10$

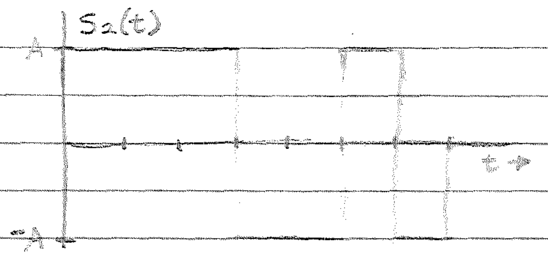


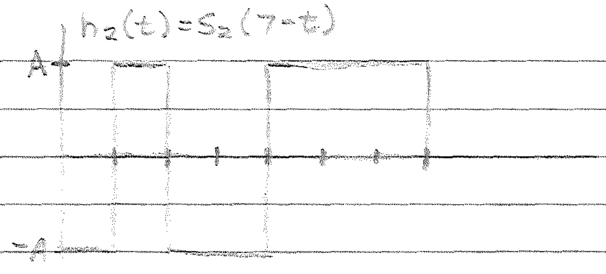
$$s_1(t)*h_1(t)=\int_{t-7}^{4}A^2d\tau=A^2(4-t+7)=A^2(-t+11)$$

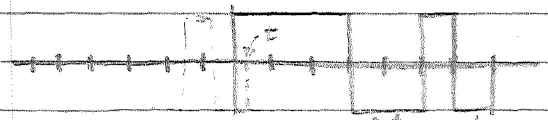
$10\le t\le 11$



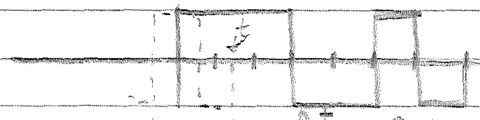
b)





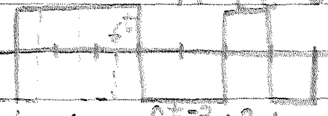


$$s_2(t) * h_2(t) = -\int_0^t A^2 d\tau = -A^2 t \qquad 0 \le t \le 1$$



$$s_2(t) * h_2(t) = \int_{t-1}^{t} -A^2 d\tau + \int_0^{t-1} A^2 d\tau$$

$$= -A^2[t - t + 1] + A^2[t-1] = +A^2[t-2] \qquad 1 \le t \le 2$$



$$s_2(t) * h_2(t) = \int_0^{t-2} -A^2 d\tau + \int_{t-2}^{t-1} A^2 d\tau + \int_{t-1}^{t} -A^2 d\tau$$

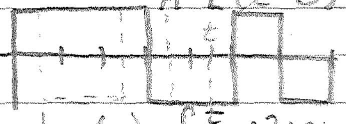
$$= A^2[(2-t) + (t-1-t+2) + (t-1-t)] = A^2(2-t)$$

$$2 \le t \le 3$$

$$s_2(t) * h_2(t) = \int_3^t +A^2 d\tau + \int_{t-1}^{3} -A^2 + \int_{t-2}^{t-1} A^2 d\tau + \int_0^{t-2} -A^2 d\tau$$

$$= A^2[(t-3) + (t-4) + (1) + (2-t)] = A^2(t-4)$$

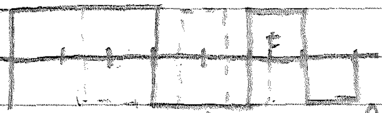
$$3 \le t \le 4$$



$$s_2(t) * h_2(t) = \int_{t-1}^{t} A^2 d\tau + \int_3^{t-1} -A^2 d\tau + \int_{t-2}^{3} A^2 d\tau + \int_{t-4}^{t-2} -A^2 d\tau + \int_0^{t-4} A^2 d\tau$$

$$= A^2[(1) + (4-t) + (5-t) + (-2) + (t-4)] = A^2[4-t]$$

$$4 \le t \le 5$$

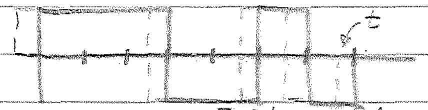


$$s_2(t) * h_2(t) = \int_5^t -A^2 d\tau + \int_{t-1}^{5} A^2 d\tau + \int_{t-2}^{t-1} -A^2 d\tau + \int_3^{t-2} A^2 d\tau + \int_{t-4}^{3} -A^2 d\tau$$

$$+ \int_0^{t-4} A^2 d\tau$$

$$= A^2[(5-t) + (6-t) + (-1) + (t-5) + (t-7) + (t-4)] = A^2(t-6)$$

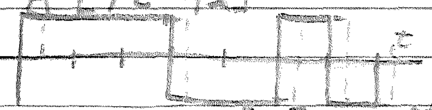
$$5 \le t \le 6$$



$s_2(t)*h_2(t)=A^2\left[\int_6^t d\tau+\int_{t-1}^6 -d\tau+\int_5^{t-1} d\tau+\int_{t-2}^5 -d\tau+\int_3^{t-2} d\tau+\int_{t-3}^3 -d\tau+\int_0^{t-4} d\tau\right]$

$=A^2\left[(t-6)+(t-7)+(t-6)+(t-7)+(t-5)+(t-7)+(t-4)\right]$

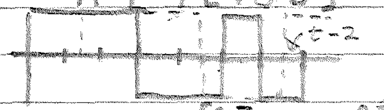
$=A^2\left[7t-42\right]$ $6\leq t\leq 7$



$s_2(t)*h_2(t)=A^2\left[\int_{t-1}^7 d\tau+\int_6^{t-1} -d\tau+\int_{t-2}^6 d\tau+\int_5^{t-2} -d\tau+\int_{t-4}^5 d\tau+\int_3^{t-4} -d\tau+\int_{t-7}^3 d\tau\right]$

$=A^2\left[(-t+8)+(-t+7)+(-t+8)+(-t+7)+(-t+9)+(-t+7)+(-t+10)\right]$

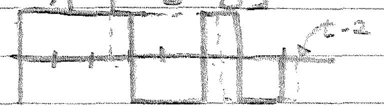
$=A^2\left[-7t+56\right]$ $7\leq t\leq 8$



$s_2(t)*h_2(t)=\left[\int_{t-2}^7 -d\tau+\int_6^{t-2} d\tau+\int_5^6 -d\tau+\int_{t-4}^5 d\tau+\int_3^{t-4} -d\tau+\int_{t-7}^3 d\tau\right]A^2$

$=A^2\left[(t-9)+(t-8)-1+(-t+9)+(-t+7)+(-t+10)\right]$

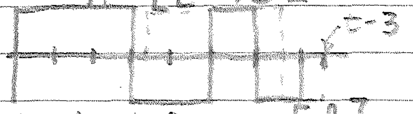
$=A^2\left[-t+8\right]$ $8\leq t\leq 9$



$s_2(t)*h_2(t)=A^2\left[\int_6^7 d\tau+\int_{t-4}^6 -d\tau+\int_5^{t-4} d\tau+\int_3^5 -d\tau+\int_{t-7}^3 d\tau\right]$

$=A^2\left[(1)+(t-10)+(t-9)-2+(-t+10)\right]$

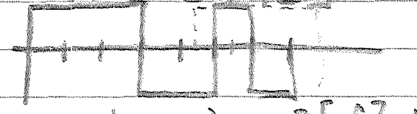
$=A^2\left[t-10\right]$ $9\leq t\leq 10$



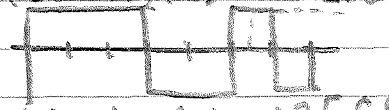
$s_2(t)*h_2(t)=A^2\left[\int_{t-4}^7 d\tau+\int_6^{t-4} -d\tau+\int_5^6 d\tau+\int_{t-7}^5 -d\tau\right]$

$=A^2\left[(-t+11)+(-t+10)+1+(t-12)\right]$

$=A^2\left[-t+10\right]$ $10\leq t\leq 11$

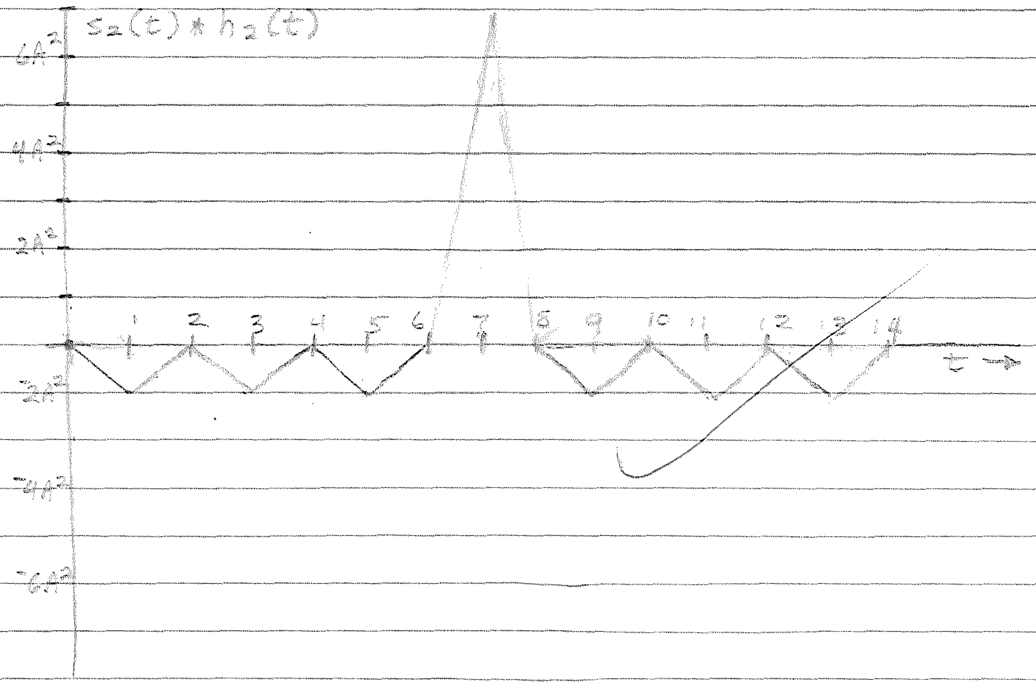


$s_2(t)*h_2(t)=A^2\left[\int_6^7 -d\tau+\int_5^6 d\tau+\int_{t-7}^5 -d\tau\right]=A^2\left[t-12\right]$ $11\leq t\leq 12$

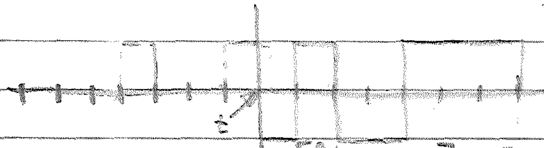


$s_2(t)*h_2(t)=A^2\left[\int_{t-7}^6 d\tau+\int_6^7 -d\tau\right]=A^2\left[12-t\right]$ $12\leq t\leq 13$

$s_2(t) * h_2(t) = A^2(t-14) \qquad 13 \le t \le 14$



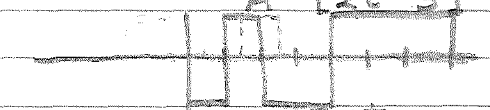
c) $s_1(t) * h_2(t)$



$s_1(t) * h_2(t) = \left[\int_0^t -d\tau\right] A^2 = A^2[-t] \qquad 0 \leq t \leq 1$

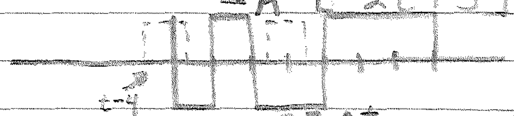
$s_1(t) * h_2(t) = A^2\left[\int_1^t d\tau + \int_{t-1}^1 -d\tau\right] = A^2\left[(t-1)+(t-2)\right]$

$= A^2[2t-3] \qquad 1 \leq t \leq 2$



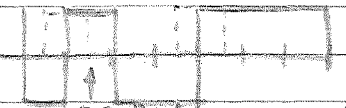
$s_1(t) * h_2(t) = A^2\left[\int_2^t d\tau + \int_{t-1}^2 d\tau\right] = A^2\left[(2-t)+(-t+3)\right]$

$= A^2[-2t+5] \qquad 2 \leq t \leq 3$



$s_1(t) * h_2(t) = A^2\left[\int_{t-1}^t -d\tau + \int_0^{t-3} -d\tau\right] = A^2[-1-t+3] = A^2[-t+2]$

$3 \leq t \leq 4$



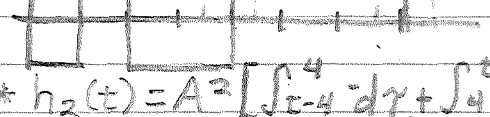
$s_1(t) * h_2(t) = A^2\left[\int_{t-4}^1 d\tau + \int_1^{t-3} d\tau + \int_{t-1}^4 d\tau + \int_4^t d\tau\right]$

$= A^2\left[(t-5)+(t-4)+(t-5)+(t-4)\right] = A^2[4t-18] \quad 4 \leq t \leq 5$

$s_1(t) * h_2(t) = A^2\left[\int_{t-4}^2 d\tau + \int_2^{t-3} -d\tau + \int_{t-1}^t d\tau\right]$

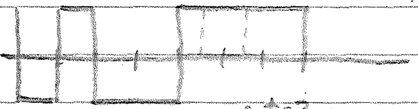
$= A^2\left[(-t+6)+(-t+5)+1\right] = A^2(-2t+12) \qquad 5 \leq t \leq 6$

$s_1(t) * h_2(t) = A^2\left[\int_{t-4}^{t-3} -d\tau + \int_{t-1}^t d\tau\right] = 0 \qquad 6 \leq t \leq 7$

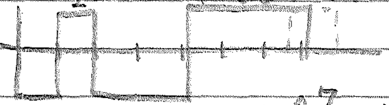


$s_1(t) * h_2(t) = A^2\left[\int_{t-4}^4 -d\tau + \int_4^{t-3} d\tau + \int_{t-1}^7 d\tau\right]$

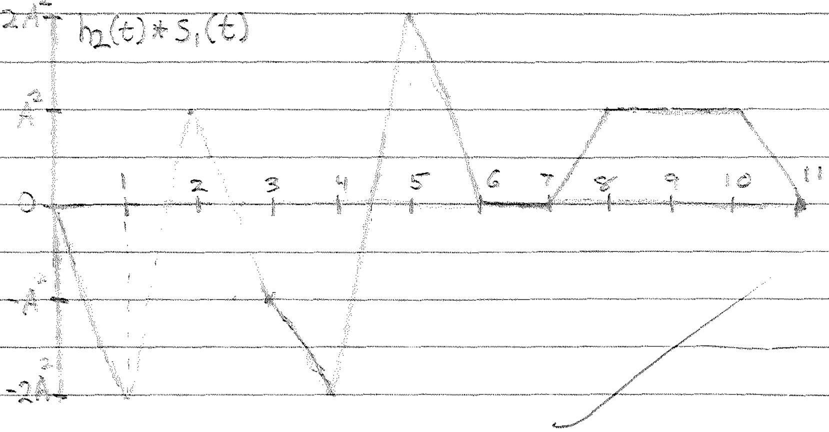
$= A^2\left[(t-8)+(t-7)+(-t+8)\right] = A^2[t-7] \quad 7 \leq t \leq 8$



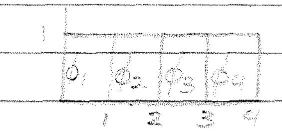
$s_1(t) * h_2(t) = A^2 \int_{t-4}^{t-3} d\tau = A^2$ $8 \le t \le 10$



$s_1(t) * h_2(t) = A^2 \int_{t-4}^{7} d\tau = A^2(-t+11)$ $10 \le t \le 11$



4-(8)



Ⓐ $S_1 = \sqrt{E/2}\,[1\,0\,0\,1]$ Ⓑ $S_1 = \sqrt{E}\,[0\,0\,0\,1]$

$S_2 = \sqrt{E/2}\,[0\,1\,1\,0]$ $S_2 = \sqrt{E}\,[0\,1\,0\,0]$

$S_3 = \sqrt{E/2}\,[1\,0\,1\,0]$ $S_3 = \sqrt{E}\,[0\,0\,1\,0]$

$S_4 = \sqrt{E/2}\,[0\,1\,0\,1]$ $S_4 = \sqrt{E}\,[1\,0\,0\,0]$

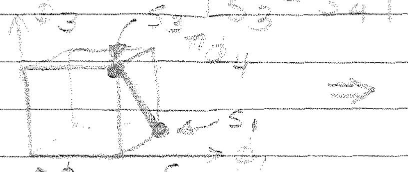
a) $\int_{-\infty}^{\infty} S_i^2(t)\,dt = E \quad \forall\, i = 1, 2, 3, 4$ IN A OR B

b) S_{1A} & S_{2A} ORTHORGONAL TO EACH OTHER

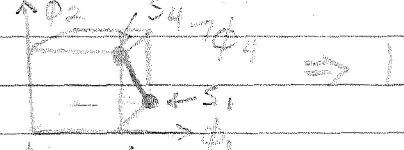
$\Rightarrow |S_2 - S_1| = \sqrt{E}$

S_{3A} & S_{4A} ORTHOGONAL TO EACH OTHER

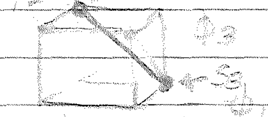
$|S_3 - S_4| = \sqrt{E}$



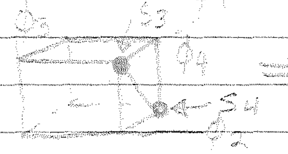
$\Rightarrow |S_3 - S_1| = \sqrt{E}$



$\Rightarrow |S_4 - S_1| = \sqrt{E_s}$



$\Rightarrow |S_3 - S_2| = \sqrt{E_s}$



$\Rightarrow |S_1 - S_2| = \sqrt{E_s}$

$$\therefore P_A[\mathcal{E}] = P_A[\mathcal{E}/m_i] = \sum_{\substack{K=0 \\ i \neq k}}^{M-1} P_2[S_i, S_K]$$

$$= \sum_{\substack{K=0 \\ i \neq k}}^{M-1} Q\left[\frac{|S_i - S_K|}{\sqrt{2N_0}}\right] = 3\,Q\left[\sqrt{\frac{E}{2N_0}}\right]$$

FOR SET B (ORTHOGONAL): $P[\mathcal{E}] = P[\mathcal{E}/m_i] \leq 3\,Q\left[\sqrt{\frac{E}{N_0}}\right]$

$P_A[\varepsilon] = 3Q\left[\sqrt{\frac{E}{2N_0}}\right]$; $P_B[\varepsilon] = 3Q\left[\sqrt{\frac{E}{4N_0}}\right]$

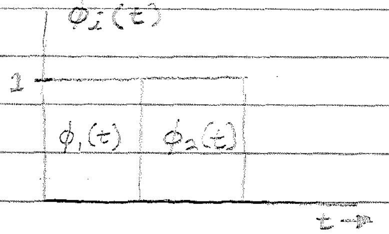
FOR SMALL $P[\varepsilon] \approx 10^{-7}$

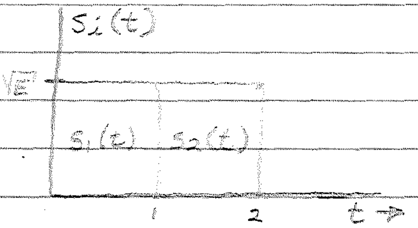
$10 \log_{10} \sqrt{\frac{2E}{N_0}} \simeq 11\ dB$; $10 \log_{10} \sqrt{\frac{E}{4N_0}} \simeq 14\ dB$

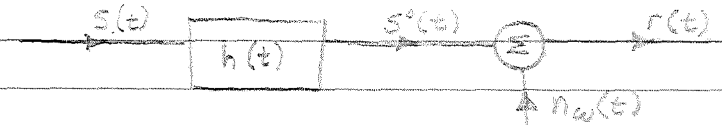
$\Rightarrow$ B COMMUNICATES ABOUT 3 dB MORE EFFECTIVELY THAN A

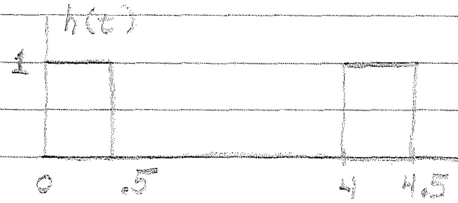
BOB MARKS

7-1)







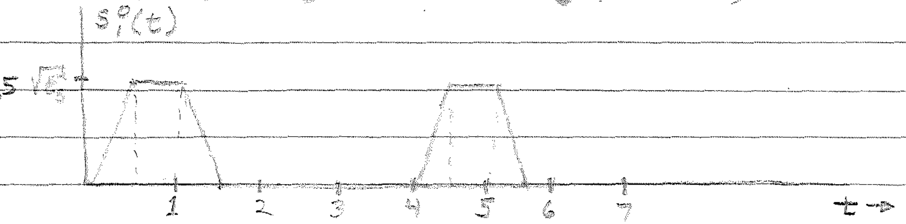


a) $s_i^o(t) = s_i(t) * h(t)$

$s_1^o(t) = \sqrt{E_s}\int_0^t dt = \sqrt{E_s}\, t \qquad 0 \le t \le .5$

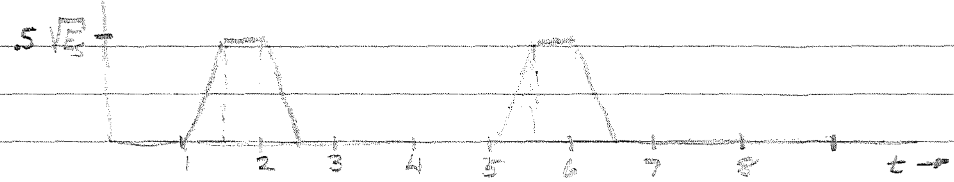
$s_1^o(t) = \sqrt{E_s}\int_0^{.5} dt = (.5)\sqrt{E_s} \qquad .5 \le t \le 1$

$s_1^o(t) = \sqrt{E_s}\int_{t-1}^{.5} dt = \sqrt{E_s}\,(1.5 - t) \qquad 1 \le t \le 1.5$

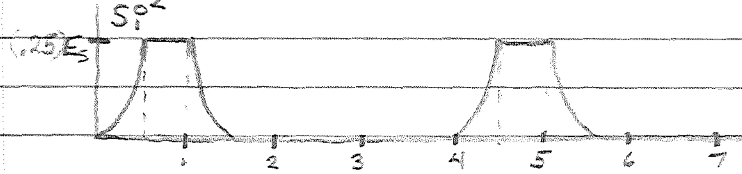


SIMILARLY

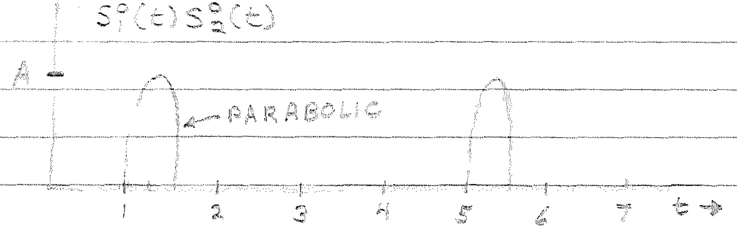
$s_2^o(t) = s_1^o(t-1)$



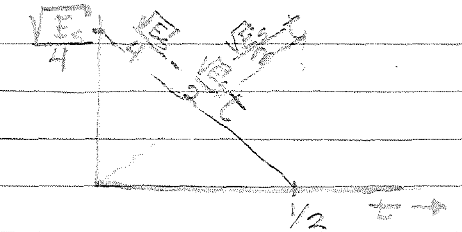
$$d^2 = \int_{-\infty}^{\infty} [s_1^o(t) - s_2^o(t)]^2 dt$$
$$= \int_{-\infty}^{\infty} [s_1^{o2}(t) - 2s_1^o(t)s_2^o(t) + s_2^{o2}(t)] dt$$



$$\int_{-\infty}^{\infty} s_1^{o2}(t)\,dt = \int_{-\infty}^{\infty} s_2^{o2}(t)\,dt$$
$$= \left(\frac{1}{4}E_s\right) + \frac{4}{3}\left(\frac{1}{4}E_s\frac{1}{2}\right)$$
$$= \frac{5}{12}E_s$$



ANALYSIS TO FIND A



$$\left(\frac{\sqrt{E_s}}{4} - \frac{\sqrt{E_s}}{2}t\right)\left(\frac{\sqrt{E_s}}{2}t\right)$$
$$= \frac{E_s}{8}t - \frac{E_s}{4}t^2$$

MAX OCCURS AT $t = \frac{1}{4}$

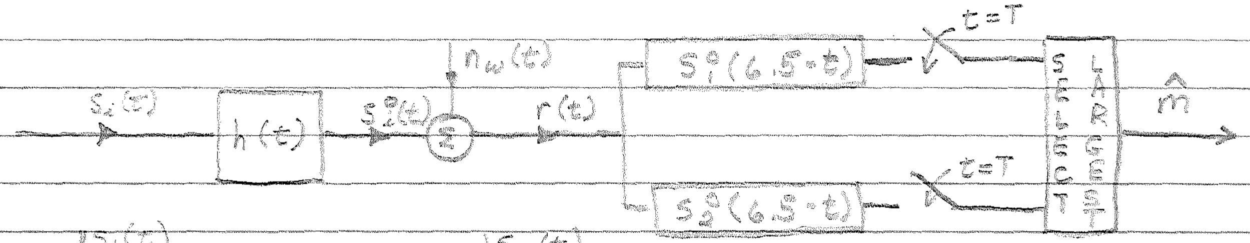
$$\Rightarrow A = E_s\left(\frac{1}{8}\cdot\frac{1}{4} - \frac{1}{4}\cdot\frac{1}{16}\right) = \frac{1}{64}E_s$$

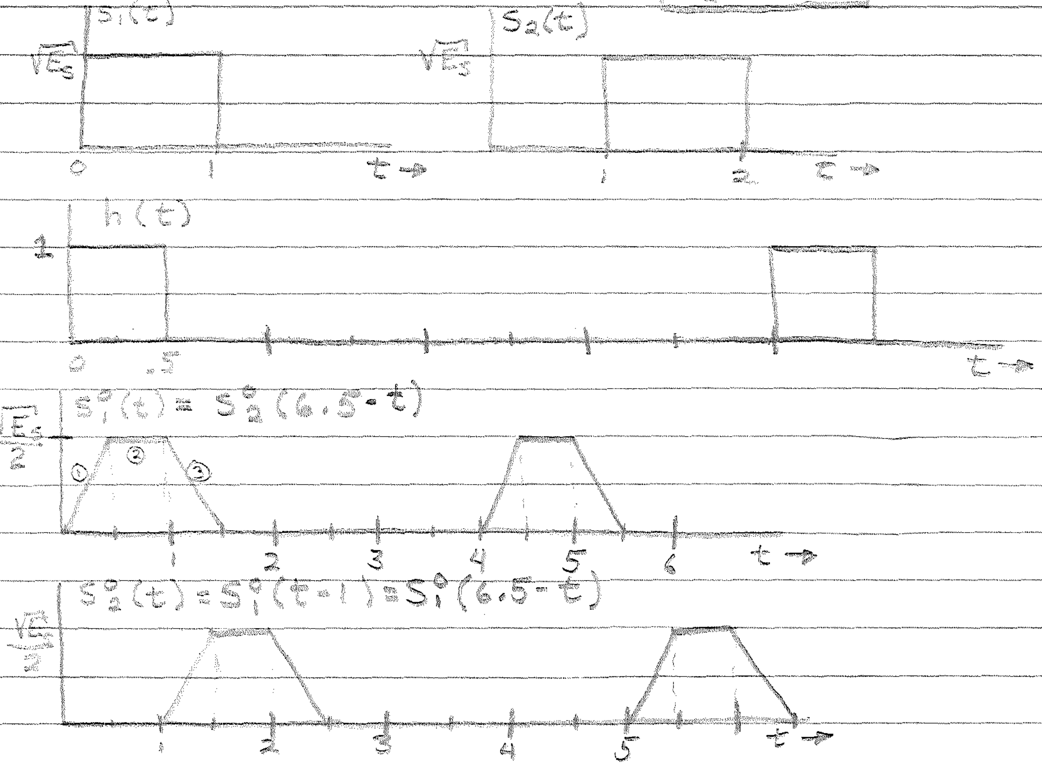
$$\Rightarrow \int_{-\infty}^{\infty} 2s_1^o(t)s_2^o(t)\,dt = 2\left[\cancel{\frac{1}{4}}\left(\frac{2}{3}\cancel{\frac{1}{4}}\frac{1}{64}E_s\right)\right] = \frac{1}{48}E_s$$

$$\therefore d^2 = (.843 - .021)E_s = .822E_s$$
$$\Rightarrow d = .908\sqrt{E_s}$$

$$P[\mathcal{E}] = Q\left[\frac{d}{\sqrt{2N_0}}\right] = Q\left[\frac{.908\sqrt{E}}{\sqrt{2N_0}}\right]$$
$$= Q\left[.908\sqrt{\frac{5}{2}}\right] = Q[1.44] = 5.6 \times 10^{-2}$$

b)





IN ABSENCE ON NOISE, OUTPUT OF $S_1^o(6.5-t)$ FILTER GIVEN $S_1(t)$ TRANSMITTED IS SAME AS $S_2^o(6.5-t)$ FILTER GIVEN $S_2(t)$ TRANSMITTED.

ie) $S_1^o(t) * S_1^o(6.5-t) = S_2^o(6.5-t) * S_2^o(t)$

ALSO, OUT OF $S_1^o(6.5-t)$ FILTER GIVEN $S_2(t)$ WAS TRANSMITTED IS $S_1^o(t) * S_1^o(6.5-t)$ SHIFTED TO THE LEFT ONE TIME UNIT

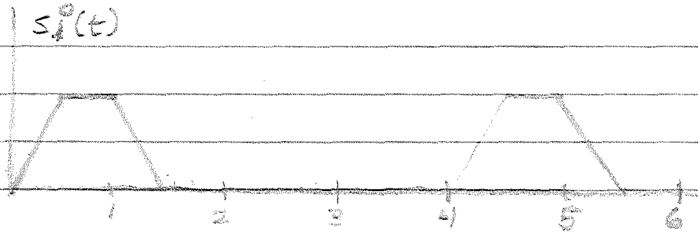
ie) GIVEN $X(t) = S_1^o(t) * S_1^o(6.5-t)$;

$S_2(t) * S_1^o(6.5-t) = X(t+1)$

SIMILARLY;

GIVEN $X(t) = S_2^o(t) * S_2^o(6.5-t)$;

$S_1(t) * S_2^o(6.5-t) = X(t-1)$



$s_1^0(t) = \sqrt{E_s}\, t \qquad 0 \le t \le \frac{1}{2}$

$s_1^0(t) = \frac{1}{2}\sqrt{E_s} \qquad \frac{1}{2} \le t \le 1$

$s_1^0(t) = \sqrt{E_s}\left(\frac{3}{2} - t\right) \qquad 1 \le t \le 3/2$

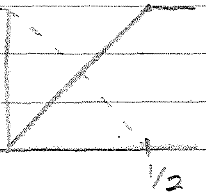
$s_1^0(t) = \sqrt{E_s}\,(t - 4) \qquad 4 \le t \le 9/2$

$s_1^0(t) = \frac{1}{2}\sqrt{E_s} \qquad 9/2 \le t \le 5$

$s_1^0(t) = \sqrt{E_s}\,(t - 1/2) \qquad 5 \le t \le 1/2$

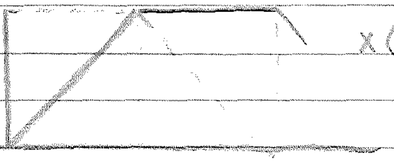
AT $t = 0$, $x(t+1) = 0$

AT $t = 1/2$



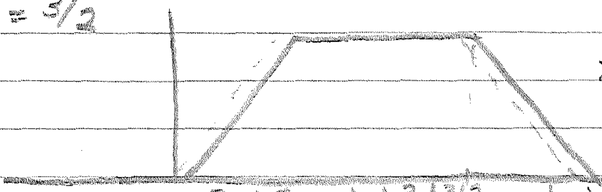
$$x(t+1) = \int_0^{1/2} \lambda\left(\frac{1}{2} - \lambda\right) d\lambda$$
$$= \frac{1}{4}\lambda^2 - \frac{1}{3}\lambda^3 \Big]_0^{1/2}$$
$$= \frac{1}{16} - \frac{1}{24} = \frac{1}{48}$$

AT $t = 1$



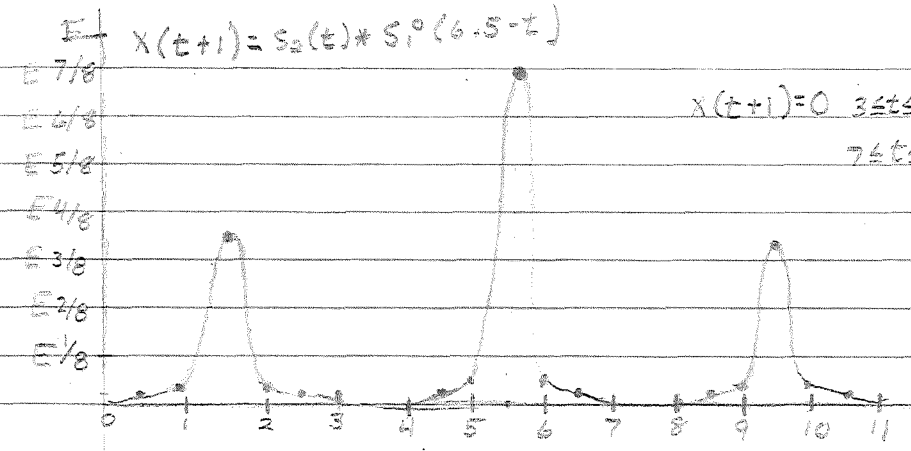
$$x(t+1) = \int_{1/2}^{1} \frac{1}{2}(1 - \lambda)\, d\lambda + \int_0^{\frac{1}{2}} \frac{1}{2}\lambda\, d\lambda$$
$$= \frac{1}{2}\lambda - \frac{1}{4}\lambda^2 \Big|_{1/2}^{1} + \frac{1}{4}\lambda^2 \Big|_0^{1/2}$$
$$= 1/32$$

AT $t = 3/2$



$$x(t+1) = \int_1^{3/2} \left(\frac{3}{2} - \lambda\right)\left(\cancelto{0}{\frac{3}{2} - \frac{3}{2}} + \lambda\right) d\lambda$$
$$+ \int_{\frac{1}{2}}^{1} \frac{1}{4}\, d\lambda$$
$$+ \int_0^{1/2} \lambda\left(\frac{3}{2} - \lambda\right) d\lambda$$
$$= \frac{3}{4}\lambda^2 - \frac{1}{3}\lambda^3 \Big|_1^{3/2} + \frac{1}{4}\lambda \Big|_{\frac{1}{2}}^{1} + \frac{3}{4}\lambda^2 - \frac{1}{3}\lambda^3 \Big|_0^{1/2}$$
$$= \frac{27}{16} - \frac{27}{24} - \frac{3}{4} + \frac{1}{3} + \frac{1}{8} + \frac{3}{16} - \frac{1}{24}$$
$$= 21/48 = 7/16$$

$X(t+1) = S_2(t) * S_1^0(6.5-t)$



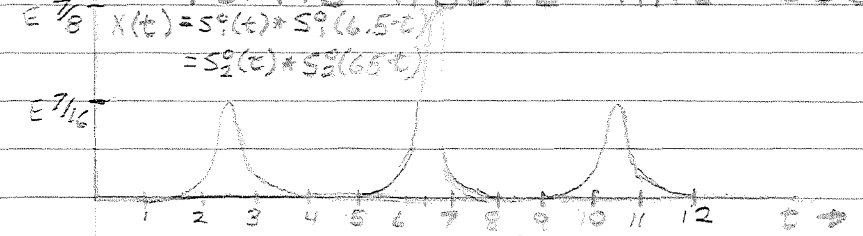
$X(t+1)=0 \quad 3 \le t \le 4$

$7 \le t \le 8$

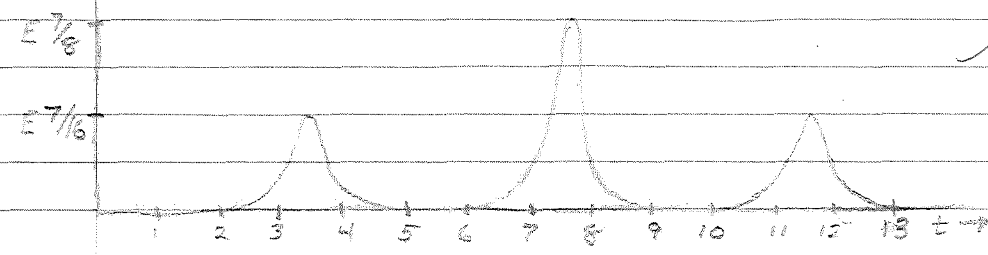
$(0,0)$	$(4,0)$
$(\frac{1}{2},\frac{1}{48})$	$(\frac{9}{2},\frac{1}{24})$
$(1,\frac{1}{32})$	$(5,\frac{1}{16})$
$(\frac{3}{2},\frac{7}{16})$	$(\frac{11}{2},\frac{7}{8})$
$(2,\frac{1}{32})$	SYMMETRIC
$(\frac{5}{2},\frac{1}{48})$	AFTERWARDS
$(3,0)$	

THE RIGHT HALF OF THE FIRST PULSE IS SYMETRIC WITH THE COMPUTED LEFT HALF. THE SECOND PULSE IS TWICE THE AMPLITUDE OF THE FIRST, AND THE LAST PULSE IS SIMILAR TO THE FIRST. ALL SEGMENTS CONNECTING DARKENED POINTS ABOVE ARE CUBICS.

$X(t) = S_1^0(t) * S_1^0(6.5-t)$
$= S_2^0(t) * S_2^0(6.5-t)$

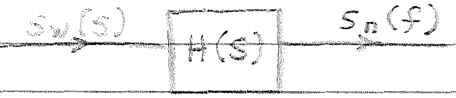


$X(t-1) = S_1(t) * S_2^0(6.5-t)$



7-2) $H(S) = \dfrac{S^3-5S^2+11S-15}{S^4+5S^3+8S^2+6S}$

a)



$$S_W(S)\,H(S)H(-S) = S_n(f) \quad ; \; S_W(S)=1$$

$$\Rightarrow S_n(S) = \frac{S^4+5S^3+8S^2+6S}{S^3-5S^2+11S-15} \cdot \frac{S^4-5S^3+8S^2-6S}{-S^3-5S^2-11S-15}$$

$$\begin{array}{r} S^2-2S+5 \\ S-3\,\overline{)\,S^3-5S^2+11S-15} \\ \underline{S^3-3S^2} \\ -2S^2+11S \\ \underline{-2S^2+6S} \\ 5S-15 \\ \underline{5S-15} \end{array} \qquad \begin{array}{r} -S^2-2S-5 \\ S+3\,\overline{)\,-S^3-5S^2-11S-15} \\ \underline{-S^3-3S^2} \\ -2S^2-11S \\ \underline{-2S^2-6S} \\ -5S-15 \\ \underline{-5S-15} \end{array}$$

$$\begin{array}{r} S^2+2S+2 \\ S+3\,\overline{)\,S^3+5S^2+8S+6} \\ \underline{S^3+3S^2} \\ 2S^2+8S \\ \underline{2S^2+6S} \\ 2S+6 \\ \underline{2S+6} \end{array} \qquad \begin{array}{r} S^2-2S+2 \\ S-3\,\overline{)\,S^3-5S^2+8S-6} \\ \underline{S^3-3S^2} \\ -2S^2+8S \\ \underline{-2S^2+6S} \\ 2S-6 \\ \underline{2S-6} \end{array}$$

$$\therefore S_n(S) = \frac{S\cancel{(S+3)}(S^2+2S+2)}{\cancel{(S-3)}(S^2-2S+5)} \cdot \frac{-S\cancel{(S-3)}(S^2-2S+2)}{\cancel{(S+3)}(S^2+2S+5)}$$

$$= \frac{S(S^2+2S+2)}{(S^2-2S+5)} \cdot \frac{-S(S^2-2S+2)}{S^2+2S+5}$$

$$= \frac{S(S+1+j)(S+1-j)\cdot{-S}(S-1+j)(S-1-j)}{(S+1-2j)(S+1+j2)(S-1-j2)(S-1+j2)}$$

b) $S_n(S) \longrightarrow \boxed{G(S)} \longrightarrow S_w(S) = 1$

$S_n(s)\, G(S)\, G(-s) = S_w(S) = 1$

$\Rightarrow G(S)\, G(-S) = \frac{(s+1-j2)(s+1+j2)}{s(s+1+j)(s+1-j)} \quad \frac{(s-1-j2)(s-1+j2)}{s(s-1+j)(s-1-j)}$

FOR REALIZABLE $G(S)$ AND $[G(S)]^{-1}$

$G(S) = \frac{(s+1-j2)(s+1+j2)}{s(s+1+j)(s+1-j)} = \frac{s^2+2s+5}{s(s^2+2s+2)}$

SINCE POLES ARE IN LEFT HALF OF S PLANE

$s \Rightarrow j2\pi f$

$\therefore G(f) = \frac{-4\pi^2 f^2 + j4\pi f + 5}{j2\pi f(-4\pi^2 f^2 + j4\pi f + 2)}$

$\frac{s^2+2s+5}{s(s^2+2s+2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+2}$

$s^2+2s+5 \equiv A(s^2+2s+2) + (Bs+C)s$

$s=0 \Rightarrow 5 = 2A \Longrightarrow A = 5/2$

$\therefore -\frac{3}{2}s^2 - 3s = (Bs+C)s$

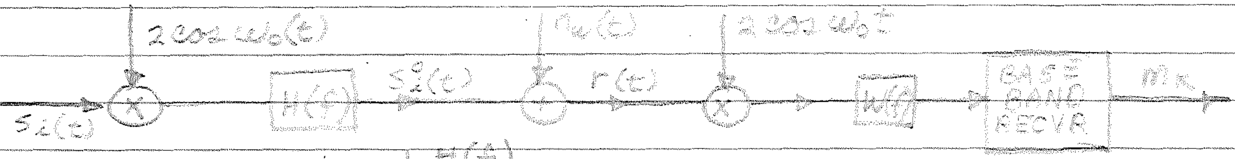
$-\frac{3}{2}s - 3 = Bs + C \Rightarrow B = -\frac{3}{2}\ ;\ C = -3$

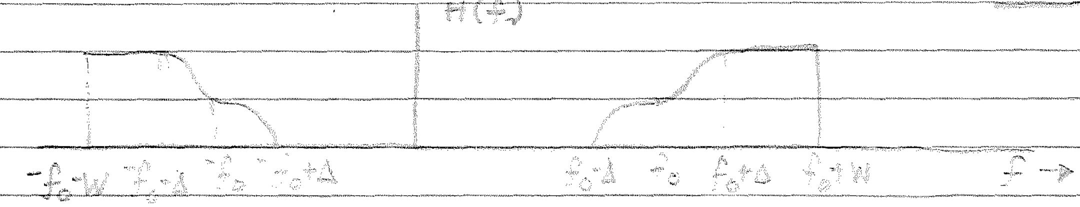
$G(S) = \frac{5/2}{s} - \frac{\frac{3}{2}s+3}{s^2+2s+2} = \frac{5/2}{s} - \frac{3}{2}\frac{(s+1)}{(s^2+2s+2)}$

$= \frac{5/2}{s} - \frac{3}{2}\frac{(s+1)}{(s+1)^2+1}$

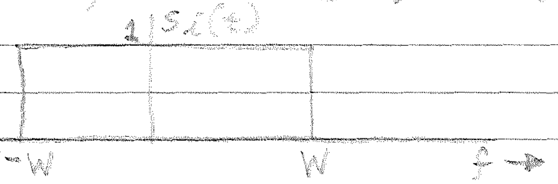
$\therefore g(t) = \left(\frac{5}{2} - \frac{3}{2}e^{-t}\cos t\right)u(t)$ ✓

7-6)



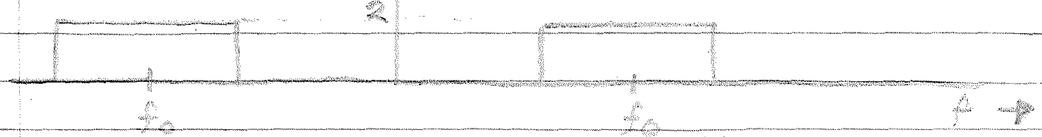


FOR ANALYSIS, LET $S_i(f) = W(f)$



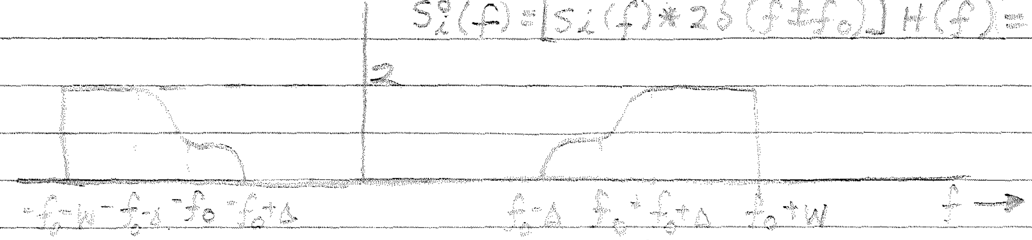
AFTER MODULATION WITH $2\cos\omega_0 t$

$S_i(f) * 2\delta(f \pm f_0)$



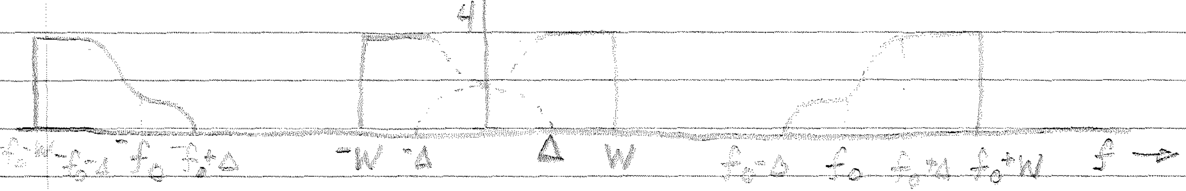
AFTER PASSING THRU $H(f)$ FILTER

$S_i^o(f) = [S_i(f) * 2\delta(f \pm f_0)]\,H(f) = H(f)$



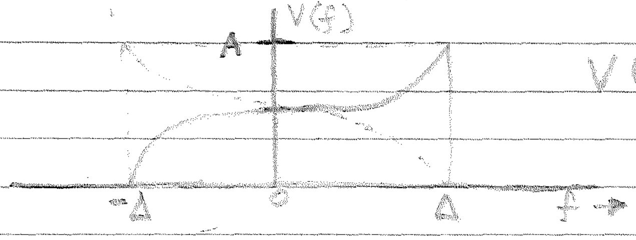
IN ABSENSE OF NOISE, $S_i^o(f)$ MODULATED WITH $2\cos\omega_0 t$

$S_i^o(f) * 2\delta(f \pm f_0)$ $(-W<f<W$ RECEIVED)



IT IS CLEAR THAT IN ORDER FOR THE RECEIVER TO OBTAIN THE AMPLIFIED VERSION OF THE TRANSMITED SIGNAL, THE GAIN OVER THE $-\Delta<f<\Delta$ INTERVAL IN THE RECEIVED SIGNAL (CONT.)

BE EQUAL TO THE CONSTANT GAIN OVER THE INTERVALS $|\Delta|<f<|W|$, IN THIS CASE, 4. THEREFORE, THE INTERVAL $-\Delta<f<\Delta$ FOR VESTIGIAL SIDEBAND MUST FOLLOW THE FOLLOWING CRITERIA:



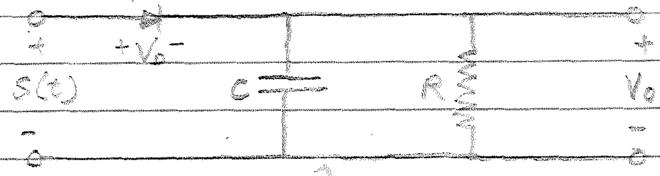
$$V(f)+V(-f)=A; \quad f<|\Delta|$$

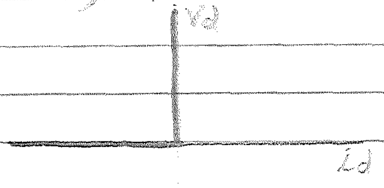
IN THAT, UNDER THESE CONDITIONS, THE RECEIVED SIGNAL IS EQUIVALENT TO THAT OF A SIMILAR SIGNAL TRANSMITTED OVER SSB OR DSB-SC, IT CAN BE ASSUMED THAT THE PROBABILITY OF ERROR IS ALSO EQUIVALENT.

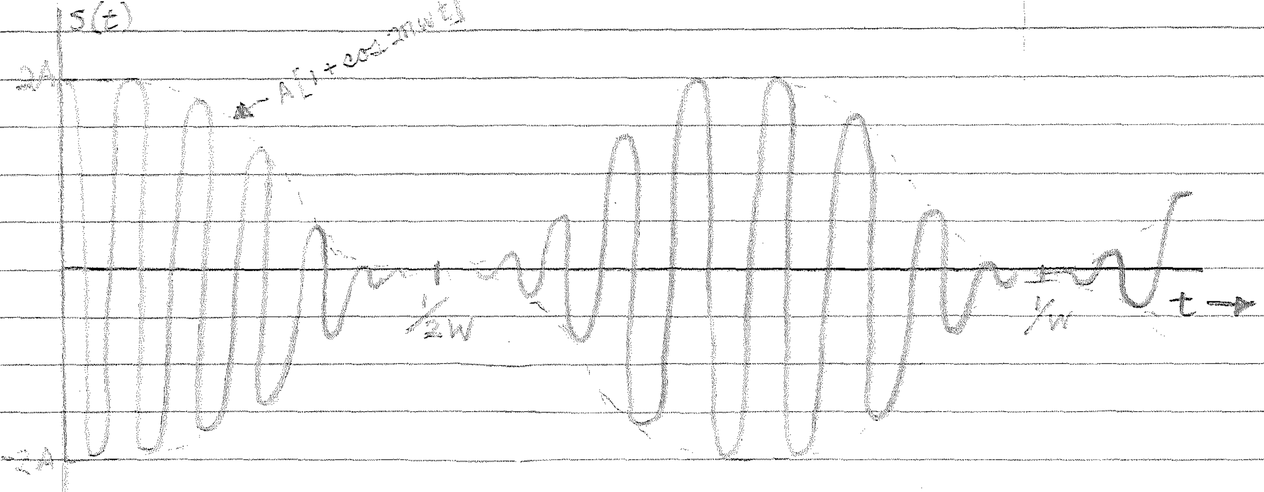
BOB MARKS

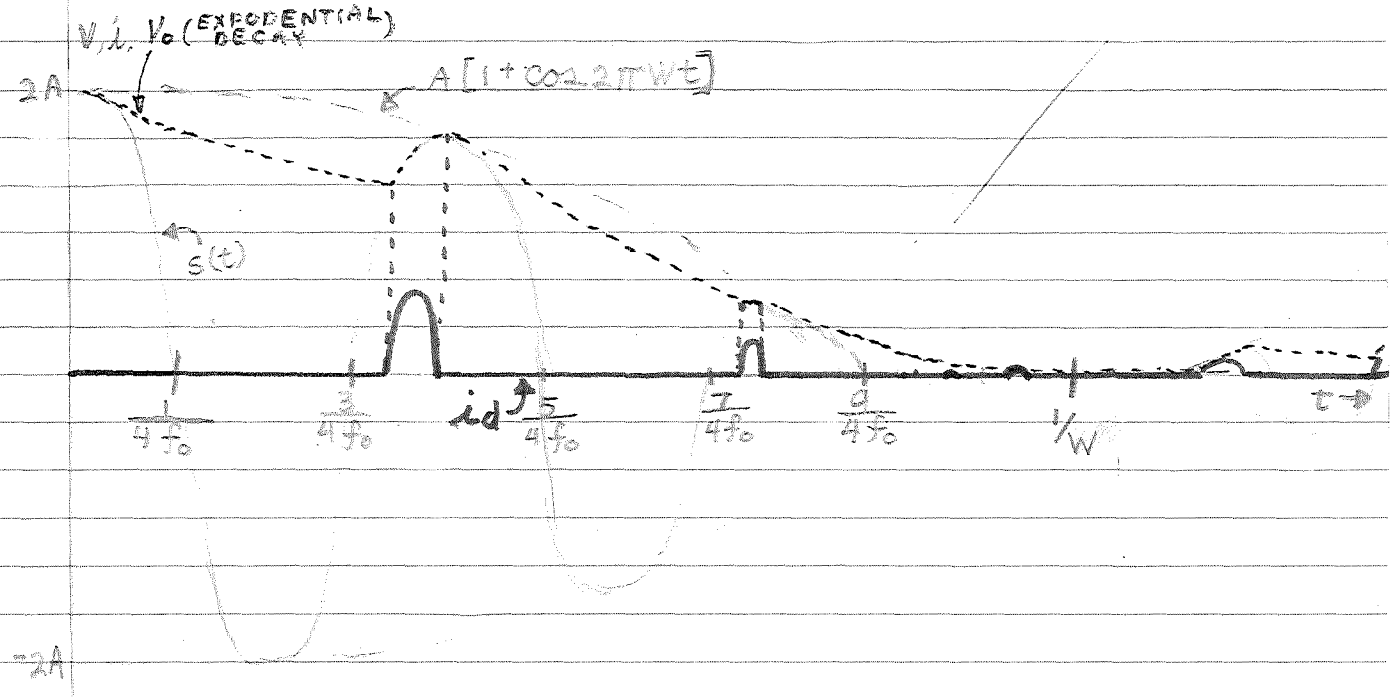
28/30

8-6) $s(t) = A[1+\cos 2\pi Wt]\cos 2\pi f_0 t$; $f_0 \gg W$









b) WISH TO CHOOSE $\tau_{MAX} = RC \ni$ THE TWO RELATIVE MAXIMA WHOSE DISTANCE FROM EACH OTHER IS MAXIMUM WILL BE CONNECTED IN SUCH A MANNER BY THE VOLTAGE FROM THE DRAINING CAPACITOR AS TO PRODUCE NO DIODE CURRENT. IF $\tau > \tau_{MAX}$, CERTAIN RELATIVE MAXIMA OF THE CARRIER WILL BE MISSED. IF $\tau < \tau_{MAX}$, ALL RELATIVE MAXIMA WILL BE UNNECESSIRALLY "UNDERSHOT." CLEARLY, BOTH OF THESE CONDITIONS YIELD A LOSS OF OPTIMALITY.

IN THAT $f_o >> W$, LINEAR ANALYSIS CLOSELY APPROXIMATES THE EXPODENTIAL DECAY.

MAXIMUM DISTANCE BETWEEN RELATIVE MAXIMA OCCURS WHEN THE SLOPE OF THE ENVELOPE IS MAX, OR

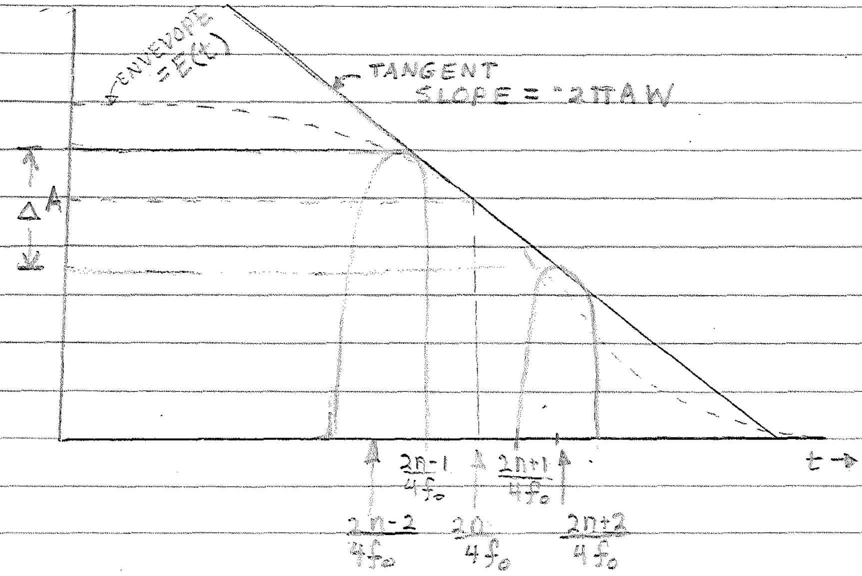
$$\frac{d^2}{dt^2}\left[A + A\cos 2\pi W t\right] = -A\,4\pi^2 W^2 \cos 2\pi W t = 0$$

OR $t = 1/2W$

SLOPE OF ENVELOPE AT $\frac{1}{2W}$:

$$\frac{d}{dt}\left[A + A\cos 2\pi W t\right]_{t=\frac{1}{2W}} = -A2\pi W$$

AGAIN, BECAUSE $f_o >> W$, ENVELOPE MAY BE REPRESENTED BY TANGENT AT POINT OF INTEREST. FURTHERMORE, CARRIER MAY BE PLACED AT CONVENIENCE, FOR SAME REASON



$$\frac{2n}{4f_0} = \frac{1}{2W} \Rightarrow n = \frac{f_0}{W}$$

$$\Delta = E\left[\frac{n-1}{2f_0}\right] - E\left[\frac{n+1}{2f_0}\right]$$

$$E(t) \simeq -2\pi AWt + C$$

$$E(t) = A \Leftrightarrow t = 1/2W$$

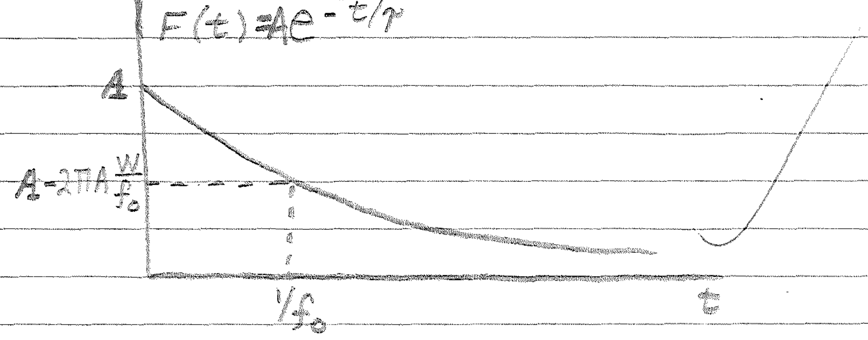
$$A \simeq -\pi A + C \Rightarrow C = A + \pi A = A(1+\pi)$$

$$\therefore E(t) \simeq -2\pi AWt + A(1+\pi)$$

$$\Rightarrow \Delta = \left|2\pi AW\left(\frac{n-1}{2f_0}\right) - 2\pi AW\left(\frac{n+1}{2f_0}\right)\right|$$

$$= 2\pi A\frac{W}{f_0}$$

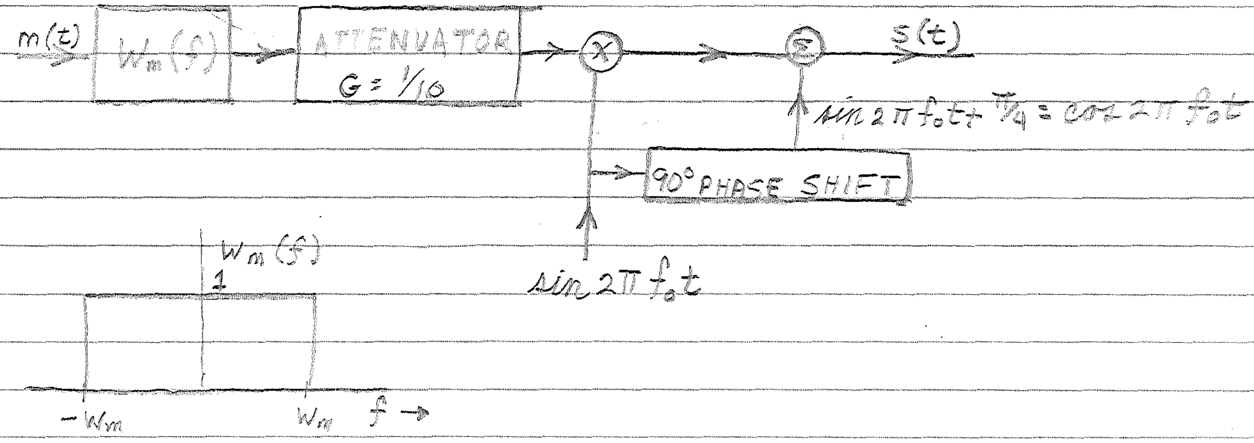
$\therefore$ THE EXPODENTIAL SHOULD DROP ABOUT $\Delta = 2\pi A\frac{W}{f_0}$ IN THE TIME INTERVAL $1/f_0$



$$A - 2\pi A\frac{W}{f_0} = e^{-1/f_0\tau} \Rightarrow \frac{-1}{f_0\tau} = \ln\left[A - 2\pi A\frac{W}{f_0}\right]$$

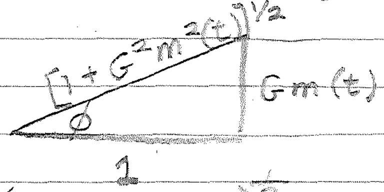
$$\text{OR } \tau_{MAX} = RC \simeq \left\{-f_0 \ln\left[A - 2\pi A\frac{W}{f_0}\right]\right\}^{-1} \simeq \left[-f_0 \ln A\right]^{-1}$$

8-16)



a) OUTPUT OF ATTENUATOR: = $Gm(t)$

$\Rightarrow s(t) = Gm(t) \sin 2\pi f_0 t + \cos 2\pi f_0 t$



$s(t) = (1+G^2m^2(t))^{\frac{1}{2}} \cos(2\pi f_0 t - \phi)$

FOR $|m(t)| \leq 1$ AND $G < 1/10$; $|Gm(t)| \leq .1$

AND $|Gm(t)|^2 \leq .01$

$\Rightarrow [1+G^2m^2(t)]^{1/2} \simeq 1$

AND $\sin Gm(t) \simeq \tan Gm(t) \cong Gm(t) \Rightarrow \phi \simeq Gm(t)$

$\therefore s(t) \simeq \cos(2\pi f_0 t - Gm(t))$

WHICH IS EQUIVALENT TO PHASE MODULATION:

$s(t) = A\sqrt{2} \cos 2\pi [f_0 t + W_2 m(t)]$ (pg 659)

WITH $-W_2 = G \leq 1/10$; $A = 1/\sqrt{2}$

b) $S_m(t)_{FM} = A\sqrt{2}\cos 2\pi\left[f_0 t + W_1 \int m(t)\,dt\right]$ (Pg 659)

$S_m(t)_{PM} = A\sqrt{2}\cos 2\pi\left[f_0 t + W_2\, m(t)\right]$

ASSUME SINUSOIDAL SIGNAL: $m(t) = \cos 2\pi W_m t$

$S_m(t)_{FM} = A\sqrt{2}\cos 2\pi\left[f_0 t + B_{FM}\cos 2\pi W_m t\right]$

$S_m(t)_{PM} = A\sqrt{2}\cos 2\pi\left[f_0 t + B_{PM}\cos 2\pi W_m t\right]$

$B_{FM} = W_1/W_m$ $\quad B_{PM} = 2\pi W_m$ (SAMPLING FREQUENCY)

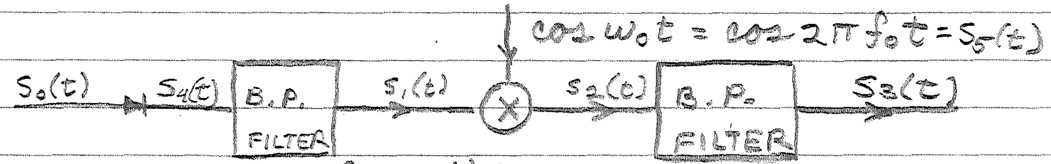
$s(t) \doteq m(t)G\sin 2\pi f_0 t + \cos 2\pi f_0 t$

$= \sqrt{1 + G^2 m^2(t)}\,\cos(2\pi f_0 t - \phi) \ni \phi = \tan^{-1} G m(t)$

$\simeq \cos(2\pi f_0 t - G m(t))$

FOR $s(t)$, $B = G = 1/10$

c)



$S_0(t) = \cos 2\pi(f_0 t + \phi),\ \ni \phi = G m(t)$

$= \cos\left[2\pi f_0 t + \beta_0 m(t)\right]$

$S_4(t) = S_0^2(t)$

$= \cos^2\left[2\pi f_0 t + B_0 m(t)\right]$

$= \frac{1}{2} + \frac{1}{2}\cos\left[2\omega_0 t + 2B_0 m(t)\right] \ni \omega_0 = 2\pi f_0$

B.P. FILTER FILTERS OUT D.C. IN $S_4(t)$

$\Rightarrow S_1(t) = \frac{1}{2}\cos\left[2\omega_0 t + 2B_0 m(t)\right]$

$S_2(t) = S_1(t)\,S_5(t)$

$= \frac{1}{2}\cos\left[2\omega_0 t + 2B_0 m(t)\right](\cos\omega_0 t)$

$= \frac{1}{4}\cos(3\omega_0 t + 2B_0 m(t)) + \frac{1}{4}\cos(\omega_0 t + 2B_0 m(t))$

SECOND B.P. FILTER, CENTERED AT f_0 $\left(= \frac{\omega_0}{2\pi}\right)$ FILTERS OUT HIGHER FREQ. COMPONENT $\cos(3\omega_0 t + 2B_0 m(t))$

$\Rightarrow S_3(t) = \frac{1}{4}\cos(\omega_0 t + 2B_0 m(t))$

$\overset{\Delta}{=} \frac{1}{4}\cos(\omega_0 t + B_1 m(t))$

$\therefore B_1 = 2B_0$

d) IF S(t) IS PASSED THRU A SECOND STAGE:

$$B_2 = 2B_1 = 2(2B_0) = 2^2 B_0$$

IF S(t) IS PASSED THRU K STAGES

$$B_K = 2B_{K-1} = 2(2B_{K-2}) = 2[2(2B_{K-3})] = 2^K B_0$$

$N_0/2 = 10^{-11}$ j ; $\overline{n^2(t)} = 10^{-4}$; $W_m = 3$ KHZ ; $P_S = 10^{-6}$ W

$$\overline{n^2(t)} = \frac{1}{(2\pi W_2)^2} \frac{N_0}{2E_S} = \frac{1}{B_K^2} \frac{N_0}{2E_S}$$

Pg. 659

$$E_S = A^2/2W_m = P_S/2W_m$$

$$10^{-4} = \frac{1}{B_K^2} \frac{10^{-11}(6\times10^3)}{10^{-6}} \Rightarrow B_K^2 = 6\times10^2$$

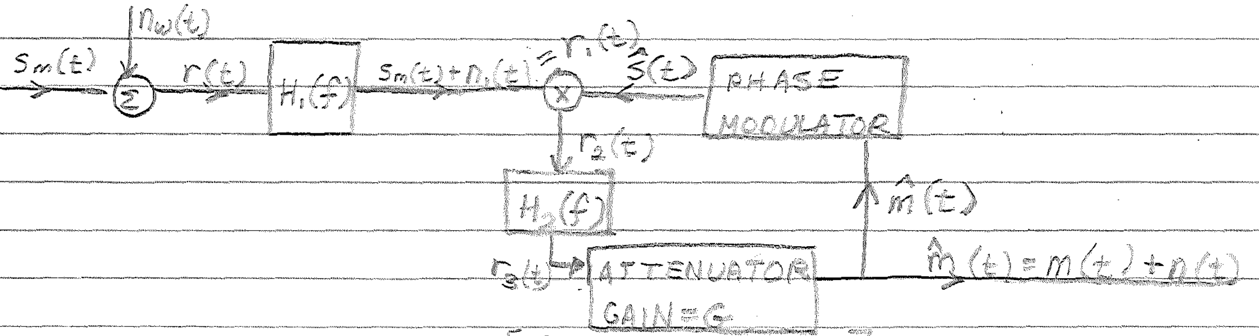
$$\therefore B_K = 24.5 = 2^K B_0$$

$$B_0 = 2\pi G = 2\pi\left(\frac{1}{10}\right) = .628 ; \frac{1}{B_0} = 1.59$$

$$\Rightarrow K = \frac{\ln B_K/B_0}{\ln 2} = \frac{\ln 39}{\ln 2} = 3.66/.693 = 5.29$$

∴ AT LEAST 6 STAGES ARE NEEDED

8-12)



$s_m(t) = A\sqrt{2}\cos 2\pi\left[f_0 t + W_2\, m(t)\right]$

$\hat{s}(t) = \sqrt{2}\cos 2\pi\left[f_0 t + \hat{W}\hat{m}(t)\right] \quad ; \hat{W} < W_2$

$r_2 = S_m(t)\,\hat{s}(t)$

$= 2A\cos 2\pi\left[f_0 t + W_2 m(t)\right]\cos 2\pi\left[f_0 t + \hat{W}\hat{m}(t)\right]$

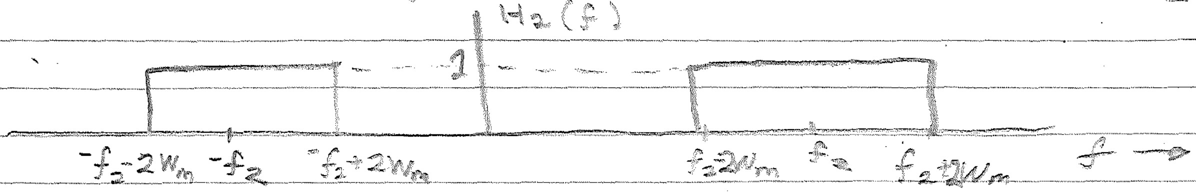
$\theta(t) \triangleq 2\pi W_2 m(t)$

$\hat{\theta}(t) \triangleq 2\pi \hat{W}\hat{m}(t)$

$\omega_0 \triangleq 2\pi f_0$

$\Rightarrow r_2(t) = A\cos\left(2\omega_0 t + \theta(t) + \hat{\theta}(t)\right) + A\cos\left(\theta(t) - \hat{\theta}(t)\right)$

$H_2(f)$ SHOULD FILTER OUT $\cos\left(2\omega_0 t + \theta(t) + \hat{\theta}(t)\right)$ CARRIER TERM, AND PASS $A\cos\left[\theta(t) - \hat{\theta}(t)\right] = r_3(t)$



$r_3(t) = A\cos\left[\theta(t) - \hat{\theta}(t)\right]$

$\hat{m}(t) = GA\cos\left[W_2 m(t) - \hat{W}\hat{m}(t)\right]$

$= GA\sin\left[W_2 m(t) - \hat{W}\hat{m}(t) + \pi/2\right]$

$-1 \ll \left[W_2\ m(t) - \hat{W}\hat{m}(t) + \frac{\pi}{2} = \psi << 1\right.$

$\Rightarrow \sin\psi = \psi$

$\therefore \hat{m}(t) = GA\left[W_2 m(t) - \hat{W}\hat{m}(t)\right]$

$\hat{m}\,(1 + GA\hat{W}) = GA\left(\frac{\pi}{2} + W_2\, m(t)\right]$

OR $\hat{m}(t) = \frac{GAW_2}{1+GA\hat{W}}\, m(t) + \frac{GA\pi}{2(1+GA)}$

PASSING $\hat{m}(t)$ THRU A DIFFERINTIATOR, AND THEN AN INTEGRATOR (WITH 0 INITIAL VALUE) FILTERS OUT CONSTANT $GA\pi/2(1+GA)$, YIELDING

$$\hat{m}(t) = \left[GAW_2 / (1+GA\hat{W})\right] m(t)$$

SETTING $\hat{m}(t) = m(t)$

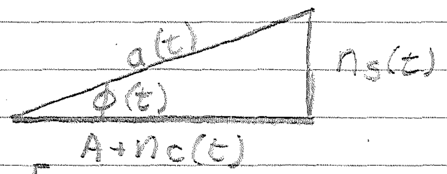
$$\Rightarrow 1+\hat{W}GA = GAW_2 \,;\, GA(W_2-\hat{W}) = 1$$

$$\therefore G = \left[A(W_2-\hat{W})\right]^{-1}$$

b) $n_1(t) = n_c(t)\sqrt{2}\cos 2\pi f_0 t + n_s(t)\sqrt{2}\sin 2\pi f_0 t$

FOR ANALYSIS, LET $m(t)=0$

$$\Rightarrow r_1(t) = [A+n_c(t)]\sqrt{2}\cos w_0 t + n_s(t)\sqrt{2}\sin w_0 t \,;\, w_0 = 2\pi f_0$$



$$a(t) = \left[n_s^2(t) + (A+n_c(t))^2\right]^{\frac{1}{2}}$$

$$n_c(t) << A >> n_s(t) \Rightarrow a(t) \simeq A$$

$$\phi(t) = \tan^{-1} -n_s(t)/[A+n_c(t)]$$

$$n_c(t) << A >> n_s(t) \Rightarrow \phi(t) \simeq -n_s(t)/[A+n_c(t)]$$

$$\Rightarrow r_1(t) = a(t)\sqrt{2}\cos[w_0 t + \phi(t)]$$

$$\simeq A\sqrt{2}\cos[w_0 t - n_s(t)/[A+n_c(t)]]$$

$$r_2(t) = r_1(t)\,\hat{s}(t)$$

$$= 2a(t)\cos[w_0 t + \phi(t)]\cos 2\pi[f_0 t + \hat{w}\hat{m}(t)]$$

$$\hat{\theta}(t) = 2\pi\hat{w}\hat{m}(t)$$

$$\Rightarrow r_2(t) = 2a(t)\cos[w_0 t + \phi(t)]\cos[2\pi f_0 t + \hat{\theta}(t)]$$

$$= a(t)[\cos(2w_0 t + \phi(t) + \hat{\theta}(t)) + \cos(\phi(t) - \hat{\theta}(t))]$$

$H_2(f)$ FILTERS OUT CARRIER COMPONENT $2w_0$

$$\Rightarrow r_3(t) = A\cos(\phi(t) - \hat{\theta}(t))$$

$$\therefore n(t) = \hat{m}(t) = GA\cos[\phi(t) - \hat{\theta}(t)] = n(t)$$

$$= GA\sin[\phi(t) - \hat{\theta}(t) + \pi/2]$$

$$\cong GA[\phi(t) - \hat{\theta}(t) + \pi/2]$$

$$= GA[-n_s(t)/A - 2\pi\hat{w}\hat{m}(t) + \pi/2]$$

2 July 1970

Solutions of 2.12, 2.23, 2.24.

(2.12). (a)

$$P[a,0] = P[0]\,P[a/0] = 0.49$$
$$P[b,0] = P[0]\,P[b/0] = 0.14.$$
$$P[c,0] = P[0]\,P[c/0] = 0.07.$$
$$P[a,1] = P[1]\,P[a/1] = 0.09$$
$$P[b,1] = P[1]\,P[b/1] = 0.06$$
$$P[c,1] = P[1]\,P[c/1] = 0.15.$$

Thus, optimum decision rule:

$$\boxed{\begin{aligned}\hat{m}(a) &= 0\\ \hat{m}(b) &= 0\\ \hat{m}(c) &= 1.\end{aligned}}$$

Prob of error = 1 − Prob of correct choice $= 1 - 0.49 - 0.14 - 0.15$
$= \boxed{0.22}$

(b) Each letter, a, b, c, is associated with a 0 or 1. Hence, we have the following 8 possibilities

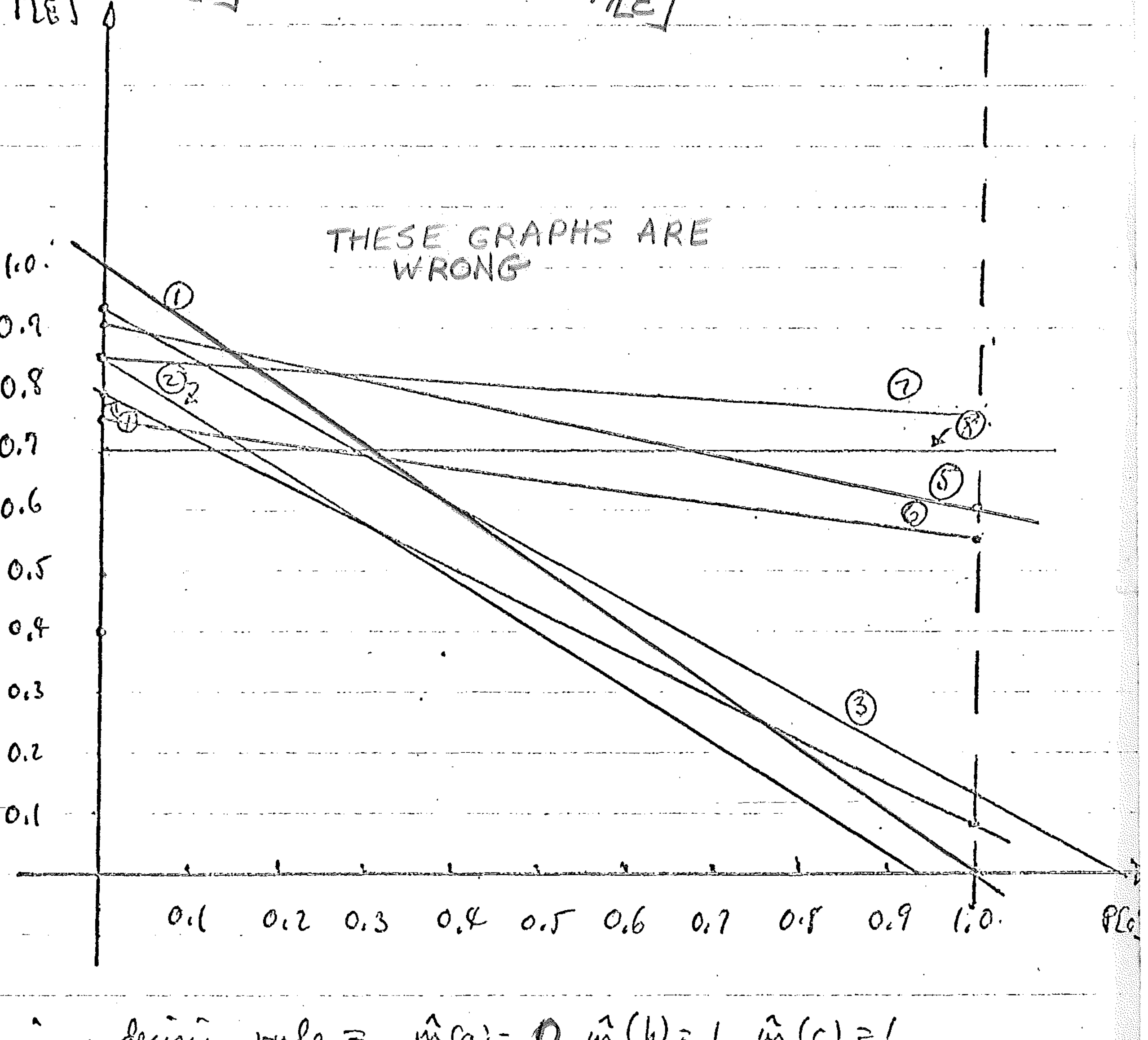
$(a, b, c) = (0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1$
① ② ③ ④ ⑤ ⑥ ⑦ ⑧

Now $P[a,0] = 0.7\,P[0].$, $P[a,1] = \cancel{0.09}. = (.3)[1 - P[0]]$
$P[b,0] = 0.2\,P[0].$, $P[b,1] = \cancel{0.06} = (.2)[1 - P[0]]$
$P[c,0] = 0.1\,P[0].$, $P[c,1] = \cancel{0.15}. = (.5)[1 - P[0]]$

$$\therefore\ P_{\textcircled{1}}[E] = P[0],\quad P_{\textcircled{2}}[E] = \overset{.9P[0]+.5[1-P[0]] \Rightarrow .5+.4P[0] = P_2(E)}{\cancel{0.9P[0]+0.15}},\quad P_{\textcircled{3}}[E] = \cancel{0.8P[0]+0.6} \Rightarrow .8-.6P[0]$$

$$P_{\textcircled{4}}[E] = \overset{.3}{\cancel{0.7}}P[0]+\cancel{0.21},\quad P_{\textcircled{5}}[E] \overset{7=1-P_4(E)}{=} 0.3P[0]+\overset{0.09}{\cancel{0.12}},\quad P_{\textcircled{6}}[E] = \overset{2+.6P[0]}{0.2P[0]+0.24} = 1-P_3[E]$$

$$P_{\textcircled{7}}[E] = \overset{.5+.4P_0}{\cancel{0.1P[0]+0.15}} = 1-P_2[E],\quad P_{\textcircled{8}}[E] = \overset{P[0]}{\cancel{0.3}} = 1-P_1[E]$$



(c) Minimax decision rule = $\hat{m}(a) = 0$, $\hat{m}(b) = 1$, $\hat{m}(c) = 1$.
All other decision rules have a larger maximum $P[E]$.
Obviously, the best decision rule is ② when $P[0] \geq 0.3$.

27 January, 1968

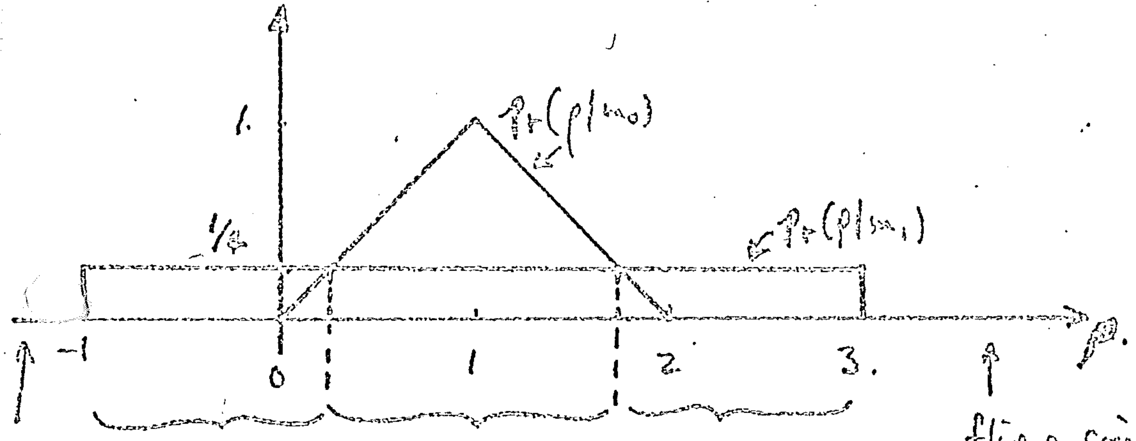
Communication Theory ; Problem Set #1 Solutions.

(2.23) Decision rule

$$\hat{m}(\rho) = m_0 \text{ iff } p_r(\rho|m_0)P[m_0] > p_r(\rho|m_1)P[m_1].$$

Because $P[m_0] = P[m_1]$ we have the maximum likelihood receiver decision rule

$$\hat{m}(\rho) = m_0 \text{ iff } p_r(\rho|m_0) > p_r(\rho|m_1).$$



flip a coin out here.

choose m_1 if ρ lies here.

Choose m_0 if ρ falls in here.

Choose m_1 if ρ lies here

flip a coin out here

Actually the probability of the received voltage lying here, or less than -1, is zero.

$$P[C|m_0] = \int_{1/4}^{7/4} p_r(\rho|m_0)\,d\rho = \frac{15}{16}$$

$$P[C|m_1] = \int_{-1}^{1/4} p_r(\rho|m_1)\,d\rho + \int_{7/4}^{3} p_r(\rho|m_1)\,d\rho = \frac{5}{8}.$$

$$P[C] = P[m_0]P[C|m_0] + P[m_1]P[C|m_1] = \frac{25}{32}$$

$$\boxed{P[\mathcal{E}] = 1 - \frac{25}{32} = 7/32}.$$

(2.24). We must calculate $P[r'=j \mid m_i]$, $i=0,1$
$j=3,2,1,0,-1,-2.$

$$P[r'=3/m_0]=P[r>2.5/m_0]=\int_{2.5}^{\infty} p_r(\alpha|m_0)\,d\alpha=0$$

$$P[r'=2/m_0]=P[1.5<r<2.5/m_0]=\int_{1.5}^{2.5} p_r(\alpha|m_0)\,d\alpha=1/8.$$

$P[r'=1\ /m_0]=3/4$

$P[r'=0/m_0]=1/8.$

$P[r'=-1/m_0]=0$

$P[r'=-2/m_0]=0.$

$$P[r'=3/m_1]=P[r>2.5/m_1]=\int_{2.5}^{\infty} p_r(\alpha|m_1)\,d\alpha=1/8.$$

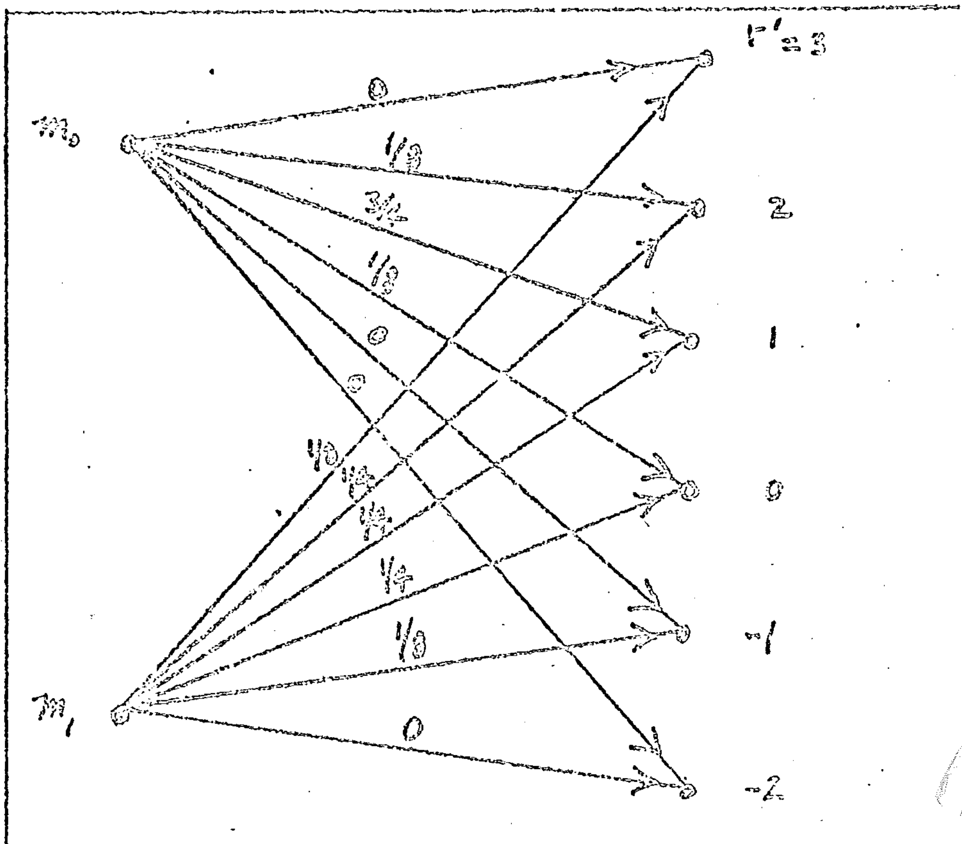
$P[r'=2/m_1]=1/4$

$P[r'=1/m_1]=1/4$

$P[r'=0/m_1]=1/4$

$P[r'=-1/m_1]=1/8.$

$P[r'=-2/m_1]=0.$



Hence, the decision rule becomes

$$\hat{m}(j) = m_i \text{ iff}$$

$$P[m_i]\,P[r_j|m_i] \geq P[m_k]P[r_j|m_k] \qquad i \neq k.$$

Since $P[m_0] = P[m_1] = 1/2$, we need only consider $P[r_j|m_0]$ and $P[r_j|m_1]$.

Thus $\hat{m}(3) = m_1$

$\hat{m}(2) = m_1$

$\hat{m}(1) = m_0$

$\hat{m}(0) = m_1$

$\hat{m}(-1) = m_1$

$\hat{m}(-2) =$ either (actually -2 cannot occur).

$$P[C|m_0] = 3/4$$

$$P[C|m_1] = 1/8 + 1/4 + 1/4 + 1/8 = 3/4.$$

$$\therefore\ P[C] = P[m_0]\,3/4 + P[m_1]\,3/4 = 3/4.$$

$\therefore\ \boxed{P[E] = 1 - 3/4 = 1/4}$ which is slightly greater than that in the unquantized system, as to be expected.

Hence, the decision rule becomes

$$\hat{m}(j) = m_i \text{ iff}$$

$$P[m_i]\, P[r_j|m_i] > P[m_k] P[r_j|m_k] \qquad i \neq k.$$

Since $P[m_0] = P[m_1] = 1/2$, we need only consider

$P[r_j|m_0]$ and $P[r_j|m_1]$.

Thus $\hat{m}(3) = m_1$

$\hat{m}(2) = m_1$

$\hat{m}(1) = m_0$

$\hat{m}(0) = m_1$

$\hat{m}(-1) = m_1$

$\hat{m}(-2) =$ either (actually -2 cannot occur).

$$P[C|m_0] = 3/4$$

$$P[C|m_1] = 1/8 + 1/4 + 1/4 + 1/8 = 3/4.$$

$$\therefore\ P[C] = P[m_0]\, 3/4 + P[m_1]\, 3/4 = 3/4.$$

$\therefore\ \boxed{P[E] = 1 - 3/4 = 1/4}$ which is slightly greater than that in the unquantized system, as to be expected.

Hence, the decision rule becomes

$$\hat{m}(j) = m_i \text{ iff}$$

$$P[m_i]\, P[r_j \mid m_i] \geq P[m_k] P[r_j \mid m_k] \qquad i \neq k.$$

Since $P[m_0] = P[m_1] = 1/2$, we need only consider $P[r_j \mid m_0]$ and $P[r_j \mid m_1]$.

Thus $\hat{m}(3) = m_1$

$\hat{m}(2) = m_1$

$\hat{m}(1) = m_0$

$\hat{m}(0) = m_1$

$\hat{m}(-1) = m_1$

$\hat{m}(-2) =$ either (actually -2 cannot occur).

$$P[C \mid m_0] = 3/4$$

$$P[C \mid m_1] = 1/8 + 1/4 + 1/4 + 1/8 = 3/4.$$

$$\therefore\ P[C] = P[m_0]\, 3/4 + P[m_1]\, 3/4 = 3/4.$$

$\therefore \boxed{P[\mathcal{E}] = 1 - 3/4 = 1/4}$ which is slightly greater than that in the unquantized system, as to be expected.

Communications Systems EE592 11 July, 1970.
Solutions of 3.4, 3.6, 3.7, 3.8.

(3.4). Since the means of the jointly Gaussian random variables X_1, X_2, and X_3 are zero, the joint density function p_{X_1,X_2,X_3} is determined by the covariances

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = R_1(0) = 1$$
$$\lambda_{12} = \overline{n_1(0)\, n_1(1)} = R_1(1) = 0$$
$$\lambda_{13} = \overline{n_1(0)\, n_2(0)} = R_{12}(0) = 1/2$$
$$\lambda_{23} = \overline{n_1(1)\, n_2(0)} = R_{12}(1) = 0.$$

Thus, X_2 is independent of both X_1 and X_3. Thus,

$$p_{X_1,X_2,X_3}(\alpha_1,\alpha_2,\alpha_3) = p_{X_2}(\alpha_2)\, p_{X_1,X_3}(\alpha_1,\alpha_3)$$
$$= \left(\frac{1}{2\pi}\right)^{3/2} e^{-\alpha_2^2/2} \cdot \frac{1}{\sqrt{1-1/4}} \exp\left[-\frac{\alpha_1^2+\alpha_3^2-2\lambda_{13}\alpha_1\alpha_3}{2(1-1/4)}\right]$$
$$= \boxed{\frac{1}{(6\pi^3)^{1/2}}\, e^{-2/3\left(\alpha_1^2+\frac{3}{4}\alpha_2^2+\alpha_3^2-\alpha_1\alpha_3\right)}}$$

3.6. $y(t) = X(t) + X(t-T)$

(a) $X(t-T)$ is Gaussian if $X(t)$ is, hence, $y(t)$ is Gaussian (sum of Gaussian r.v. is Gaussian).

(b) $m_y(t) = \overline{y(t)} = \overline{X(t) + X(t-T)} = \overline{X(t)} + \overline{X(t-T)}$

$$= \boxed{m_X(t) + m_X(t-T)}$$

$$\mathcal{L}_X(t,s) = E\{(X(t) - m_X(t))(X(s) - m_X(s))\}.$$

$$= R_X(t,s) - m_X(t)\, m_X(s).$$

$$\mathcal{L}_y(t,s) = R_y(t,s) - m_y(t)\, m_y(s)$$

$$= E\{y(t)y(s)\} - m_y(t)\, m_y(s)$$

$$= E\{X(t)X(s) + X(t-T)X(s) + X(t)X(s-T) + X(t-T)X(s-T)\}$$

$$- m_y(t)\, m_y(s)$$

$$= \boxed{\mathcal{L}_X(t,s) + \mathcal{L}_X(t-T,s) + \mathcal{L}_X(t,s-T) + \mathcal{L}_X(t-T,s-T)}$$

(c) Yes, y(t) is stationary if x(t) is stationary because then the mean of y(t) is constant and the covariance depends on $t-s=\tau$.

(3.7). (a) $y_i(t) = \int_{-\infty}^{\infty} X(\alpha)\, h_i(t-\alpha)\, d\alpha$.

$$\overline{y_i(t)} = \int_{-\infty}^{\infty} \overline{X(\alpha)}\, h_i(t-\alpha)\, d\alpha$$

$$= 0 \quad , \quad \forall i.$$

$$\overline{y_i^2(t)} = R_{y_i}(0)$$

$$R_{y_i}(\tau) = \iint_{-\infty}^{\infty} R_X(\alpha,\beta)\, h_i(t+\tau-\alpha)\, h_i(t-\beta)\, d\alpha\, d\beta.$$

(Eq. 3.108).

$$= \iint_{-\infty}^{\infty} \underset{\text{white noise}}{\frac{N_0}{2}}\, \delta(\alpha-\beta)\, h_i(t+\tau-\alpha)\, h_i(t-\beta)\, d\alpha\, d\beta$$

$$= \frac{N_0}{2} \int_{-\infty}^{\infty} h_i(t+\tau-\beta)\, h_i(t-\beta)\, d\beta.$$

Thus $\overline{y_i^2(t)} = R_{y_i}(\overset{\tau=0}{0}) = \frac{N_0}{2}\int_{-\infty}^{\infty} h_i^2(t-\beta)\, d\beta.$

$$\therefore \boxed{\overline{y_1^2(t)} = \frac{N_0}{2}\int_{t-1}^{t} 1\cdot d\beta = \frac{N_0}{2}.}$$

$$\boxed{\overline{y_2^2(t)} = \frac{N_0}{2}\int_{-\infty}^{t} 4e^{2(\beta-t)}\, d\beta = N_0}$$

$$\overline{y_3^2(t)} = \frac{N_0}{2}\int_{t-2}^{t} 2\sin^2 2\pi(\beta-t)\, d\beta.$$

$$= N_0\int_{t-2}^{t}\left(\frac{1}{2} - \frac{1}{2}\cos 4\pi\,(\beta-t)\right) d\beta.$$

$$= N_0 - \frac{N_0}{8\pi}\sin 4\pi(\beta-t)\Big|_{t-2}^{t} =$$

$$= \boxed{N_0}$$

(6) The cross-correlation function $E\{y_i(t)\,y_j(s)\}$ is given by

$$R_{ij}(t,s) = \iint_{-\infty}^{\infty} R_X(\alpha,\beta)\, h_i(t-\alpha)\, h_j(s-\beta)\, d\alpha\, d\beta.$$

$$= \iint_{-\infty}^{\infty} \frac{N_0}{2}\, \delta(\alpha-\beta)\, h_i(t-\alpha)\, h_j(s-\beta)\, d\alpha\, d\beta.$$

$$= \frac{N_0}{2} \int_{-\infty}^{\infty} h_i(t-\beta)\, h_j(s-\beta)\, d\beta.$$

Now $R_{ij}(t,t) = \frac{N_0}{2} \int_{-\infty}^{\infty} h_i(t-\beta)\, h_j(t-\beta)\, d\beta.$

Let $\rho = t-\beta.$, then

$$R_{ij}(t,t) = \frac{N_0}{2} \int_{-\infty}^{\infty} h_i(\rho)\, h_j(\rho)\, d\rho.$$

$R_{13}(t,t) = \frac{N_0}{2} \int_0^1 1\cdot\sqrt{2}\sin 2\pi\rho\, d\rho$

$= 0 \quad \forall t.$ This is the only pair for which this holds for all t.

(c). $$R_{ij}(t,s) = \frac{N_0}{2}\int_{-\infty}^{\infty} h_i(\xi)\, h_j(\xi-\tau)\, d\xi, \quad \tau = t-s.$$

$$= R_{ij}(\tau).$$

The only pair of interest (which could possibly give $R_{ij}(\tau)=0$ $\forall\tau$) is $i=1, j=3$, again.

$$\therefore R_{13}(\tau) = \frac{N_0}{2}\int_0^1 \sqrt{2}\sin 2\pi(\xi-\tau)\, d\xi.$$

$$= -\frac{N_0}{2}\cdot\frac{\sqrt{2}}{2\pi}\cos 2\pi(\xi-\tau)\Big|_0^1$$

$$= -\frac{N_0}{2\sqrt{2}\pi}\left[\cos 2\pi(1-\tau) - \cos 2\pi\tau\right].$$

$$= \frac{-N_0}{2\sqrt{2}\pi}\left(\cos 2\pi\cos 2\pi\tau + \sin 2\pi \sin 2\pi\tau - \cos 2\pi\tau\right)$$

$$= \boxed{0 \quad \forall\tau.}$$

(3.8). $$y(t) = \int_{-\infty}^{\infty} X(\alpha)\, h(t-\alpha)\, d\alpha.$$

$$z(t) = \int_{-\infty}^{\infty} X(\beta)\, g(t-\beta)\, d\beta.$$

(a) $m_{y_1} = \int_{-\infty}^{\infty} m_x\, h(t_1-\alpha)\,d\alpha = m_x \int_{-\infty}^{\infty} h(t_1-\alpha)\,d\alpha.$

$m_{z_2} = m_x \int_{-\infty}^{\infty} g(t_2-\beta)\,d\beta.$

$$\sigma_1^2 = E[(y_1-m_{y_1})^2] = E[y_1^2] - m_{y_1}^2$$

$$= E\left[\iint_{-\infty}^{\infty} x(\alpha)\,x(\beta)\, h(t_1-\alpha)\,h(t_1-\beta)\,d\alpha\, d\beta\right] - m_{y_1}^2$$

$$= \iint_{-\infty}^{\infty} R_X(\alpha-\beta)\, h(t_1-\alpha)h(t_1-\beta)\,d\alpha\, d\beta - m_{y_1}^2$$

$$= \iint_{-\infty}^{\infty} (\mathcal{L}_X(\alpha-\beta) + m_X^2)\; h(t_1-\alpha)\,h(t_1-\beta)\,d\alpha\, d\beta - m_{y_1}^2.$$

$$= \iint_{-\infty}^{\infty} (\mathcal{L}_X(\alpha-\beta) + m_X^2)\, h(t_1-\alpha)\,h(t_1-\beta)\,d\alpha\, d\beta - m_X^2 \iint_{-\infty}^{\infty} h(t_1-\alpha)\, h(t_1-\beta)\, d\alpha$$

$$= \iint_{-\infty}^{\infty} \mathcal{L}_X(\alpha-\beta) \overset{h(t_1-\alpha)h(t_1-\beta).}{\swarrow} d\alpha\, d\beta$$

$$\sigma_2^2 = \iint_{-\infty}^{\infty} \mathcal{L}_X(\alpha-\beta)\, g(t_2-\alpha)\, g(t_2-\beta)\, d\alpha\, d\beta.$$

$$\rho_{12} = \frac{E[y_1 z_2]}{\sigma_1\sigma_2} - \frac{m_{y_1} m_{z_2}}{\sigma_1\sigma_2}.$$

$$= \frac{1}{\sigma_1\sigma_2}\Big[\iint_{-\infty}^{\infty} R_x(\alpha-\beta)\, h(t_1-\alpha)\, g(t_2-\beta)\, d\alpha\, d\beta.$$

$$- m_{y_1} m_{z_2}\Big].$$

$$= \frac{1}{\sigma_1\sigma_2}\Big[\iint_{-\infty}^{\infty} \mathcal{R}_x(\alpha-\beta)\, h(t_1-\alpha)\, g(t_2-\beta)\, d\alpha\, d\beta\Big].$$

Thus $p_{y_1,z_2}(\varrho_1,\varrho_2) = \dfrac{1}{2\pi\sigma_1\sigma_2(1-\rho_{12}^2)^{1/2}} \times$

$$\times \exp\left\{-\frac{1}{2(1-\rho_{12}^2)}\left[\frac{(\varrho_1-m_{y_1})^2}{\sigma_1^2} - \frac{2\rho_{12}(\varrho_1-m_{y_1})(\varrho_2-m_{z_2})}{\sigma_1\sigma_2}\right.\right.$$

$$\left.\left. + \frac{(\varrho_2-m_{z_2})^2}{\sigma_2^2}\right]\right\}, \quad \text{(Eqn. 3.47).}$$

with $\rho_{12}, \sigma_1, \sigma_2, m_{y_1}, m_{z_2}$ defined as above (see also 3.78).

(b) $$y(t) = \int_{-\infty}^{\infty} x(\alpha)\, h(t-\alpha)\, d\alpha$$

$$\overline{y(t)\,x(t-\tau)} = \int_{-\infty}^{\infty} \overline{x(\alpha)\,x(t-\tau)}\; h(t-\alpha)\, d\alpha$$

$$= \int_{-\infty}^{\infty} R_x(\alpha-(t-\tau))\, h(t-\alpha)\, d\alpha.$$

If $x(t)$ is white noise, then $R_x(\alpha-(t-\tau)) = \frac{N_0}{2}\,\delta(\alpha-(t-\tau))$

and $$\overline{y(t)\,x(t-\tau)} = \frac{N_0}{2}\int_{-\infty}^{\infty} \delta(\alpha-(t-\tau))\, h(t-\alpha)\, d\alpha$$

$$= \frac{N_0}{2}\, h(\tau).$$

Thus, the cross-correlation of the output of a linear filter with white noise input yields a means of "identifying the system", i.e., computing the system impulse response.

(c) $$E[y(t_1)\,z(t_2)] = \iint_{-\infty}^{\infty} R_x(\alpha-\beta)\, h(t_1-\alpha)\, g(t_2-\beta)\, d\alpha\, d\beta$$

must ~~[illegible]~~ satisfy

$$E[y(t_1)\,z(t_2)] - m_1(t_1)\, m_2(t_2) = L_{yz}(t_1,t_2) = 0.$$

$$m_{y_1} m_{z_2} = m_x^2 \iint_{-\infty}^{\infty} h(t_1-\alpha)\, g(t_2-\beta)\, d\alpha\, d\beta .$$

$$\iint_{-\infty}^{\infty} R_x(\alpha-\beta)\, h(t_1-\alpha)\, g(t_2-\beta)\, d\alpha\, d\beta$$

$$= \iint_{-\infty}^{\infty} \mathcal{L}_x(\alpha-\beta)\, h(t_1-\alpha)\, g(t_2-\beta)\, d\alpha\, d\beta + m_x^2 \iint_{-\infty}^{\infty} h(t_1-\alpha)\, g(t_2-\beta)\, d\alpha\, d\beta .$$

$$\therefore \boxed{\mathcal{L}_{yz}(t_1,t_2) = \iint_{-\infty}^{\infty} \mathcal{L}_x(\alpha-\beta)\, h(t_1-\alpha)\, g(t_2-\beta)\, d\alpha\, d\beta = 0}$$

for statistical independence of $y(t_1), z(t_2)$, $\forall\, t_1, t_2$.

(d). $$\overline{y^2(t)} = \int_{-\infty}^{\infty} S_y(f)\, df = \int_{-\infty}^{\infty} S_x(f)\, |H(f)|^2\, df .$$

$$S_x(f) = \mathcal{F}\{R_x(\tau)\} = \mathcal{F}\{\mathcal{L}_x(\tau)\} \quad (\text{if } m_x = 0).$$

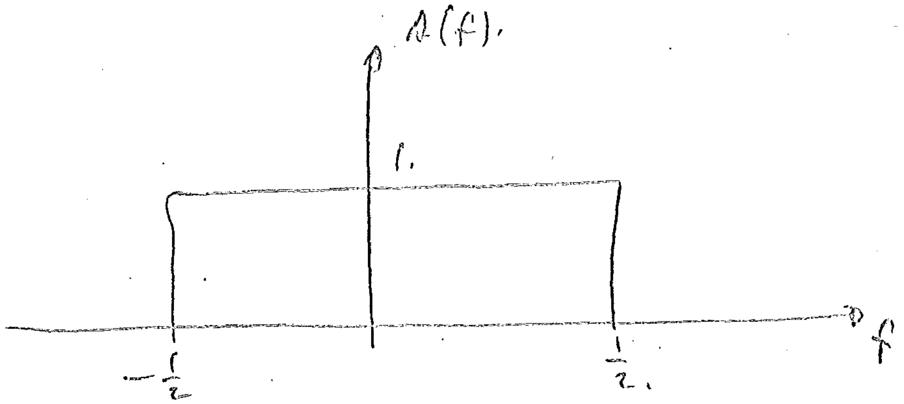
$$= \mathcal{F}\left\{\left(\frac{\sin \pi\tau}{\pi\tau}\right)^2\right\} = \mathcal{F}\left\{\frac{\sin \pi\tau}{\pi\tau}\right\} * \mathcal{F}\left\{\frac{\sin \pi\tau}{\pi\tau}\right\}.$$

convolution in f space.

$$= \int_{-\infty}^{\infty} A(f')\, A(f-f')\, df', \quad \text{where } A(f) = \mathcal{F}\left\{\frac{\sin \pi\tau}{\pi\tau}\right\}$$

What is $A(f)$?

$$A(f) = \int_{-\infty}^{\infty} e^{-j2\pi f\tau} \frac{\sin \pi\tau}{\pi\tau} d\tau = \begin{matrix} 1, & -\frac{1}{2} \le f \le \frac{1}{2} \\ 0, & \text{elsewhere} \end{matrix}$$



$$\text{Thus } S_x(f) = \int_{-\infty}^{\infty} A(f') A(f-f') df' = \int_{-\frac{1}{2}}^{\frac{1}{2}} A(f-f') df'$$

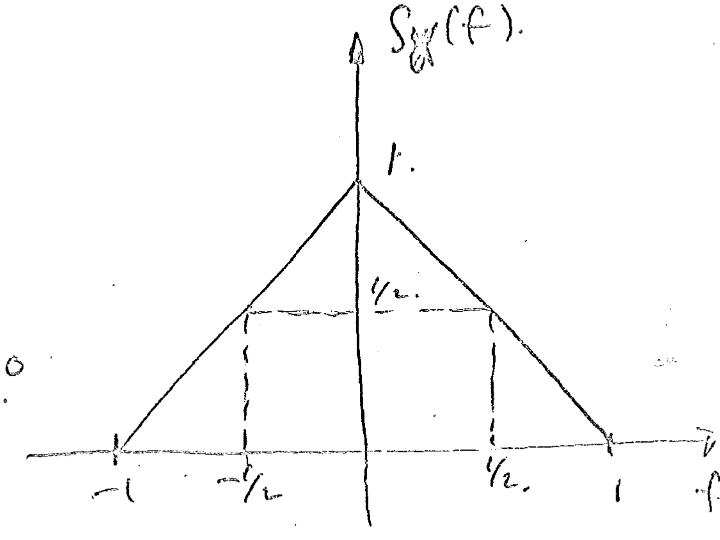
$$= \int_{f-\frac{1}{2}}^{f+\frac{1}{2}} A(\xi) d\xi \quad (\text{for } \xi = f - f').$$

$= 0 , \quad f < -1.$

$= f + 1 , \quad -1 < f < 0$

$= 1 - f , \quad 0 < f < 1.$

$= 0 , \quad 1 < f$



⑫

$$\therefore \quad \overline{y^2(t)} = \int_{-\infty}^{\infty} S_x(f)|H(f)|^2 df = \int_{-1}^{-1/2} S_x(f)\,df + \int_{1/2}^{1} S_x(f)\,df$$

$$= \boxed{\frac{1}{4}}$$

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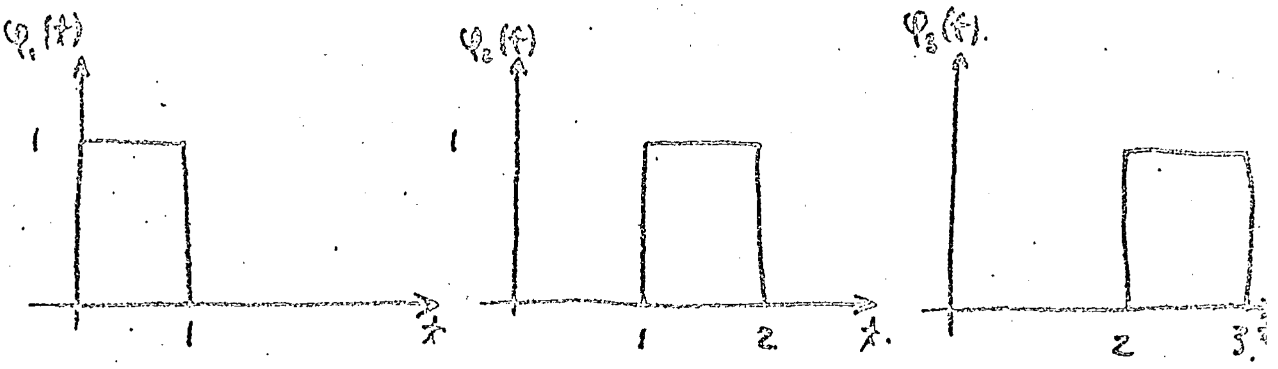
(1) (7.9).



clearly: $S_1(t) = \varphi_1(t) \Rightarrow \bar{S}_1 = [1,0,0]$.

$S_2(t) = \varphi_2(t) \Rightarrow \bar{S}_2 = [0,1,0]$

$S_3(t) = \varphi_1(t)+\varphi_2(t) \Rightarrow \bar{S}_3 = [1,1,0]$.

$S_4(t) = \varphi_3(t) \Rightarrow \bar{S}_4 = [0,0,1]$.

$S_5(t) = \varphi(t)+\varphi_3(t) \Rightarrow \bar{S}_5 = [1,0,1]$.

$S_6(t) = \varphi_2(t)+\varphi_3(t) \Rightarrow \bar{S}_6 = [0,1,1]$.

$S_7(t) = \varphi_1(t)+\varphi_2(t)+\varphi_3(t) \Rightarrow \bar{S}_7 = [1,1,1]$.

$S_8(t) = 0 \Rightarrow \bar{S}_0 = [0,0,0]$.

$\int_{-\infty}^{\infty} \varphi_i(t)\varphi_j(t)\,dt = \delta_{ij}$. Incidentally, note the similarity to three-place binary numbers.

$\bar{n} = [n_1, n_2, n_3]$

$n_1 = \int_{-\infty}^{\infty} n_\omega(t)\,\varphi_1(t)\,dt$, $n_2 = \int_{\infty} n_\omega(t)\,\varphi_2(t)\,dt$, $n_3 = \int_{-\infty}^{\infty} n_\omega(t)\,\varphi_3(t)\,dt$.

$\bar{n}_1 = \bar{n}_2 = \bar{n}_3 = 0$. Because $\varphi_1(t)$, $\varphi_2(t)$, $\varphi_3(t)$ are orthogonal we know from our treatment of prob. (7.7) that n_1, n_2, and n_3 are statistically independent r.v.'s and that $p_{\bar{n}}(n_1, n_2, n_3) = \frac{1}{\left(2\pi \frac{N_0}{2}\right)^{3/2}} e^{-(n_1^2+n_2^2+n_3^2)/N_0}$.

Because the messages are equally likely, our decision function is $p_{\bar{r}}(\bar{\rho} \mid \bar{s} = \bar{s}_i) = p_{\bar{n}}(\bar{\rho} - \bar{s}_i)$. Thus

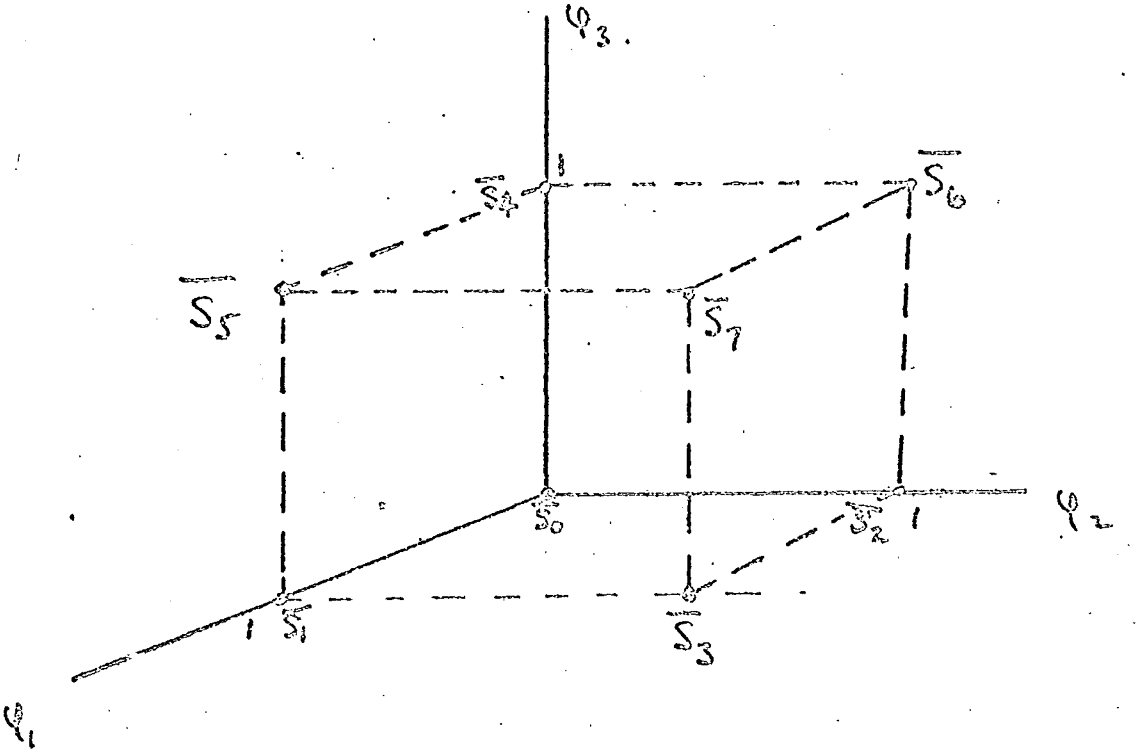
$$\hat{m}(\bar{\rho}) = m_i \text{ iff } p_{\bar{n}}(\bar{\rho} - \bar{s}_i) > p_{\bar{n}}(\bar{\rho} - \bar{s}_j).$$

Our i^{th} decision interval is

$$I_i : \text{set of } \bar{\rho} \ni p_{\bar{n}}(\bar{\rho} - \bar{s}_i) > p_{\bar{n}}(\bar{\rho} - \bar{s}_j).$$

$$P[C \mid m_i] = \int_{I_i} p_{\bar{n}}(\bar{\rho} - \bar{s}_i).d\bar{\rho}$$

$$P[\mathcal{E}] = 1 - P[C] = 1 - \frac{1}{8} \sum_i \int_{I_i} p_{\bar{n}}(\bar{\rho} - \bar{s}_i)\, d\bar{\rho}.$$



The boundaries of the decision regions are the perpendicular bisectors of the lines connecting the various lattice points $(\bar{S}_0, \cdots, \bar{S}_7)$.

$I_0 = -\infty < \rho_1 \le 1/2 \;, \; -\infty < \rho_2 \le 1/2 \;, \; -\infty < \rho_3 \le 1/2.$

$I_1 = 1/2 \le \rho_1 < \infty \;, \; -\infty < \rho_2 \le 1/2, \; -\infty < \rho_3 \le 1/2.$

$I_2 = -\infty < \rho_1 \le 1/2 \;,$ ~~$-\infty < \rho_2 \le 1/2$~~ $\; -\infty < \rho_3 \le 1/2.$

$1/2 \le \rho_2 < \infty$

$I_3 = \frac{1}{2} \le \rho_1 < \infty \;, \; 1/2 \le \rho_2 < \infty \;, \; -\infty < \rho_3 \le 1/2.$

$I_4 = -\infty < \rho_1 \le 1/2 \;, \; -\infty < \rho_2 \le 1/2, \; 1/2 \le \rho_3 < \infty$

$I_5 = 1/2 \le \rho_1 < \infty \;, \; -\infty < \rho_2 \le 1/2, \; 1/2 \le \rho_3 < \infty.$

$I_6 =$ ~~$1/2 \le \rho_1 < \infty$~~ $\; 1/2 \le \rho_2 < \infty, \; 1/2 \le \rho_3 < \infty.$

$-\infty < \rho_1 \le 1/2.$

$I_7 = 1/2 \le \rho_1 < \infty \;, \; 1/2 \le \rho_2 < \infty, \; 1/2 \le \rho_3 < \infty.$

$$\int_{I_0} p_{\bar{n}}(\bar{\rho}-\bar{s}_0) = \frac{1}{(\pi N_0)^{3/2}} \int_{-\infty}^{1/2} e^{-[\rho_1^2+\rho_2^2+\rho_3^2]/N_0} \, d\rho_1 \int_{-\infty}^{1/2} d\rho_2 \int_{-\infty}^{1/2} d\rho_3.$$

$$= \left[\left(\frac{1}{\pi N_0}\right)^{1/2} \int_{-\infty}^{1/2} e^{-\rho^2/N_0} \, d\rho \right]^3 = \left[1 - Q\left(\frac{1}{\sqrt{2N_0}}\right)\right]^3 = Q^3\left\{\frac{-1}{\sqrt{2N_0}}\right\}$$

$$\int_{I_1} p_{\bar{n}}(\bar{\rho}-\bar{s}_1) = \left(\frac{1}{\pi N_0}\right)^{3/2} \int_{1/2}^{\infty} e^{-(\rho_1-1)^2/N_0} \, d\rho_1 \cdot \int_{-\infty}^{1/2} e^{-\rho_2^2/N_0} \, d\rho_2 \int_{-\infty}^{1/2} e^{-\rho_3^2/N_0} \, d\rho_3.$$

$$= Q^3\left[\frac{-1}{\sqrt{2N_0}}\right].$$

$\vdots$

$$\int_{I_7} p_{\bar{n}}(\bar{\rho}-\bar{s}_7) = \left(\frac{1}{\pi N_0}\right)^{3/2} \int_{1/2}^{\infty} e^{-(\rho_1-1)^2/N_0} \, d\rho_1 \int_{1/2}^{\infty} e^{-(\rho_2-1)^2/N_0} \, d\rho_2 \int_{1/2}^{\infty} e^{-(\rho_3-1)^2/N_0} \, d\rho_3$$

$$= Q^3\left[\frac{-1}{\sqrt{2N_0}}\right].$$

$$\therefore \boxed{P[\mathcal{E}] = 1 - \frac{1}{8}\cdot 8\, Q^3\left[\frac{-1}{\sqrt{2N_0}}\right] = 1 - Q^3\left[\frac{-1}{\sqrt{2N_0}}\right].}$$

$$= 1 - \left[1 - Q\left(\frac{1}{\sqrt{2N_0}}\right)\right]^3$$

(2). (4.13). (a) If $m = m_0$, then

$$r(t) = \alpha_1 s_0(t-\tau_1) + n_1(t) + \alpha_2 s_0(t-\tau_2) + n_2(t) + \alpha_3 s_0(t-\tau_3) + n_3(t).$$

If $m = m_1$, then

$$r(t) = -\alpha_1 s_0(t-\tau_1) + n_1(t) - \alpha_2 s_0(t-\tau_2) + n_2(t) - \alpha_3 s_0(t-\tau_3) + n_3(t).$$

Let $S_a(t) \equiv \alpha_1 S_0(t-\tau_1) + \alpha_2 S(t-\tau_2) + \alpha_3 S(t-\tau_3)$

$-S_a(t) = -(\alpha_1 S_0(t-\tau_1) + \alpha_2 S(t-\tau_2) + \alpha_3 S(t-\tau_3))$.

We would sketch these two functions but the results ~~[illegible]~~ aren't worth it.

Define $\|S_a(t)\| \equiv \int_{-\infty}^{\infty} S_a^2(t)dt$. Then the G-S orthogonalization process gives us $\varphi(t) = \frac{S_a(t)}{\|S_a(t)\|} \triangleq \frac{S_a(t)}{A}$. Hence, the one-dimensional vector representation is:

$m_0 \Leftrightarrow S_a(t) \Leftrightarrow S_0 = A.$

$m_1 \Leftrightarrow -S_a(t) \Leftrightarrow S_1 = -A.$

$$n = \int_{\infty}^{\infty}\varphi(t)[n_1(t)+n_2(t)+n_3(t)]dt = \int_{-\infty}^{\infty}\varphi(t)n_1(t)dt + \int_{-\infty}^{\infty}\varphi(t)n_2(t)dt + \int_{-\infty}^{\infty}\varphi(t)n_3(t)dt$$

$= n_1 + n_2 + n_3$. n_1, n_2, n_3 are Gaussian r.v.'s with mean 0 and variance: $\overline{n_1^2} = 0.002$, $\overline{n_2^2} = 0.006$, $\overline{n_3^2} = 0.004$. n, being the sum of Gaussian r.v.'s is also Gaussian. $\overline{n} = \overline{n_1} + \overline{n_2} + \overline{n_3} = 0$.

$$\overline{n^2} = \overline{n_1^2} + \overline{n_2^2} + \overline{n_3^2} + 2\overline{n_1 n_2} + 2\overline{n_1 n_3} + 2\overline{n_2 n_3} = \overline{n_1^2} + \overline{n_2^2} + \overline{n_3^2} = 0.012$$

because $\overline{n_1 n_2}$, $\overline{n_1 n_3}$ and $\overline{n_2 n_3} = 0$ (independent r.v.'s).

$$\therefore \quad p_n(\alpha) = \frac{1}{\sqrt{2\pi(0.012)}}\, e^{-\alpha^2/2(0.012)}$$

$$r = \int_{-\infty}^{\infty} r(t)\varphi(t)\,dt \quad , \quad s_0 = \int_{-\infty}^{\infty} S_a(t)\varphi(t)\,dt \quad , \quad s_1 = \int_{-\infty}^{\infty} -S_a(t)\varphi(t)\,dt$$

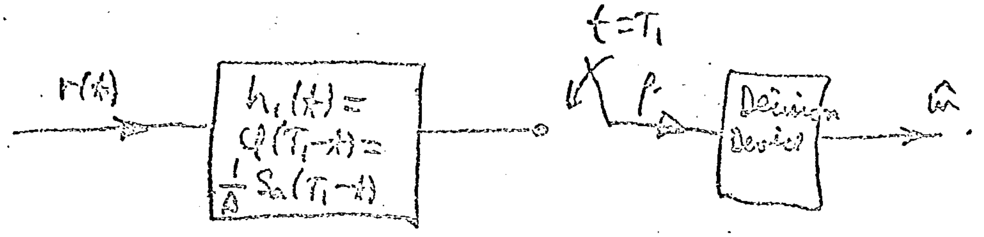
$\therefore$ $r = s_{0,1} + n$. Because the messages are equally likely the decision ~~function~~ becomes $p_r(\rho \mid s = s_i) = p_n(\rho - s_i)$.

Thus, $\hat{m}(\rho) = m_0$ iff

$$\frac{1}{\sqrt{2\pi(0.012)}}\, e^{-(\rho-A)^2/2(0.012)} > \frac{1}{\sqrt{2\pi(0.012)}}\, e^{-(\rho+A)^2/2(0.012)}$$

Thus, $\hat{m}(\rho) = m_0$ iff $\rho > 0$
$\hat{m}(\rho) = m_1$ iff $\rho < 0$.

But ρ is a particular value of $\int_{-\infty}^{\infty} r(t)\,\varphi(t)\,dt = \int_{-\infty}^{\infty} r(\tau)\varphi(T_1 - t + \tau)\,d\tau$ when $t = T_1$.



The decision device simply determines the sign of ρ and sets
$\hat{m}(\rho) = m_0$ iff $\rho > 0$
$\hat{m}(\rho) = m_1$ iff $\rho < 0$.

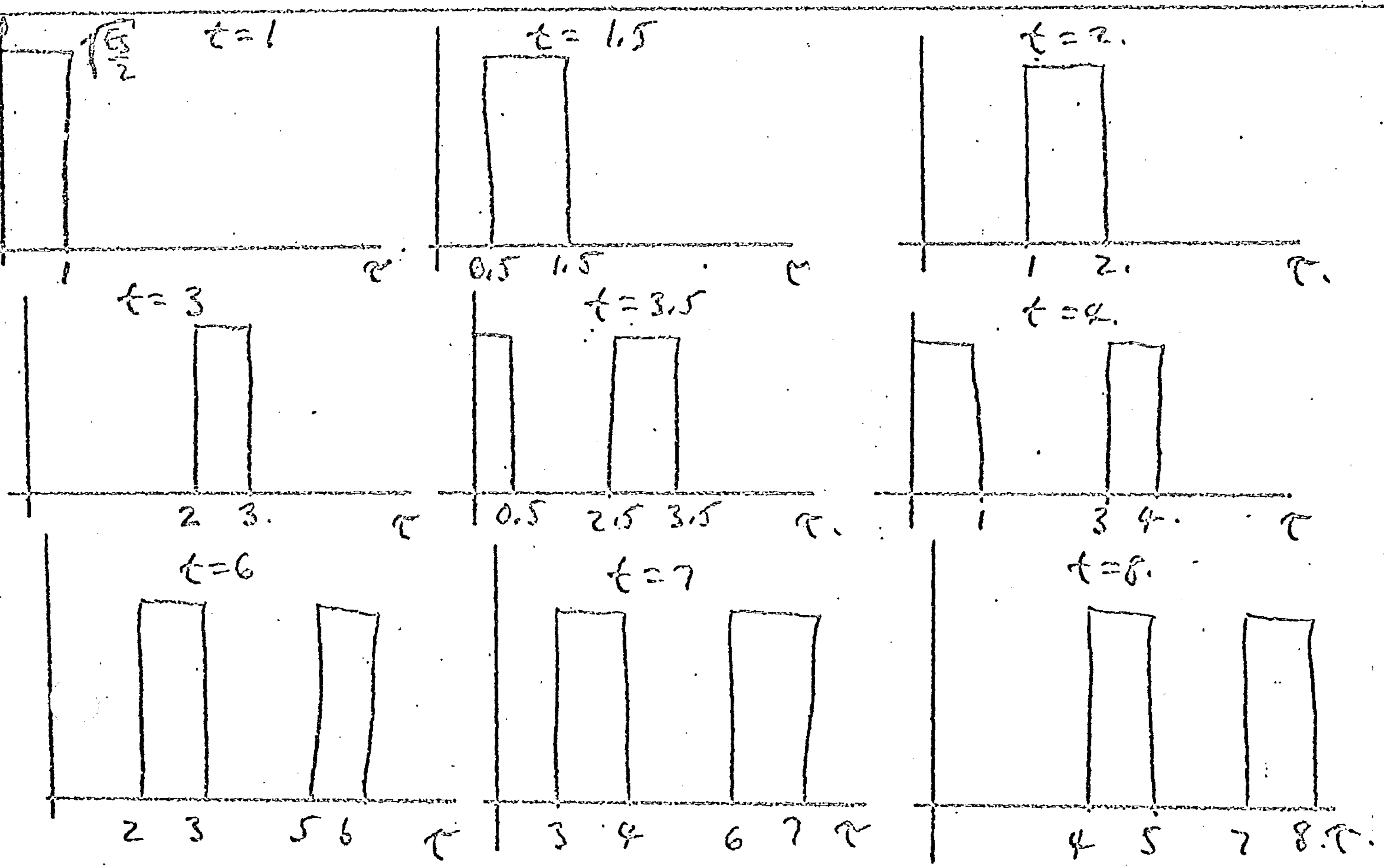
$T_1 = \overset{5}{\cancel{7.5}} \times 10^{-3}$ sec.

$$P[\mathcal{E}] = 1 - \frac{1}{2}\cdot 2\int_0^{\infty} \frac{1}{\sqrt{2\pi(0.012)}}\, e^{-(\rho-A)^2/2(0.012)}\, d\rho = 1 - Q\left(\frac{-A}{\sqrt{0.012}}\right).$$

$A \equiv \|S_a(t)\| = \int_{-\infty}^{\infty} S_a^2(t)\,dt$ may be determined easily and $P[\mathcal{E}]$ computed.

(3) (4.14). Match the input at $T = 4$ sec. Then $h(t) = S_1(4-t)$.

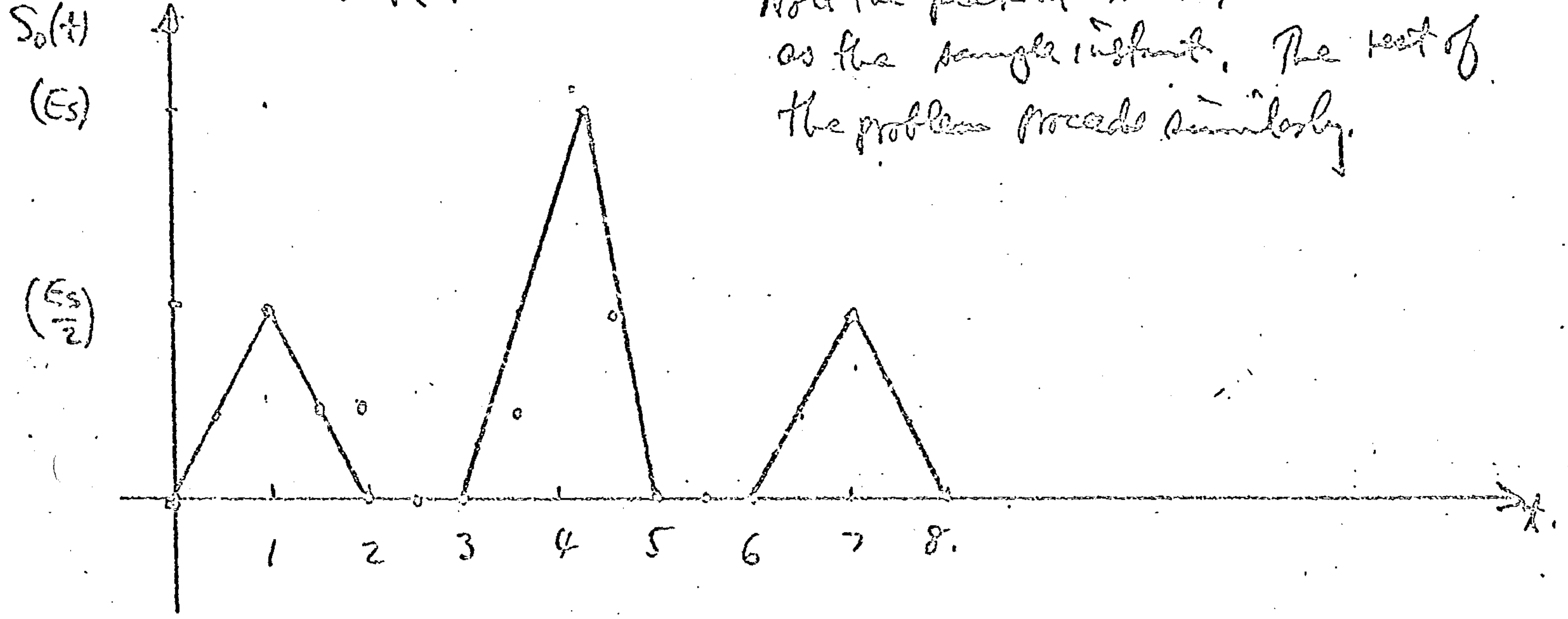
$S_0(t) = \int_{\infty}^{\infty} S_1(4-t+\tau)\, S_1(\tau)\, d\tau$. We sketch $S_1(4-t+\tau)$ for various values of t.



Hence, the area under the product curve for various values of t looks like this.

like:

Note the peak at t=4, which was taken as the sample instant. The rest of the problem proceeds similarly.



(4). (4.18). (a) The area under the square of each signal is the same for each set, namely E_s.

(b) The union bound depends on the ratio $\frac{|\bar{S}_i-\bar{S}_k|}{\sqrt{2N_0}}$ (eqn. 4.110).

For the signals of set (b), $\frac{|\bar{S}_i-\bar{S}_k|}{\sqrt{2N_0}} = \sqrt{\frac{E_s}{N_0}}$ for every pair i-k.

Hence, the average is also $\sqrt{\frac{E_s}{N_0}}$.

For the signals of set (a): $\frac{|\bar{S}_i-\bar{S}_k|}{\sqrt{2N_0}} = \sqrt{\frac{E_s}{N_0}}$ for i-k = 1-2, 3-

$\frac{|\bar{S}_i-\bar{S}_k|}{\sqrt{2N_0}} = \sqrt{\frac{E_s}{2N_0}}$ for i-k = 1-3, 1-4, 2-3, 2-4.

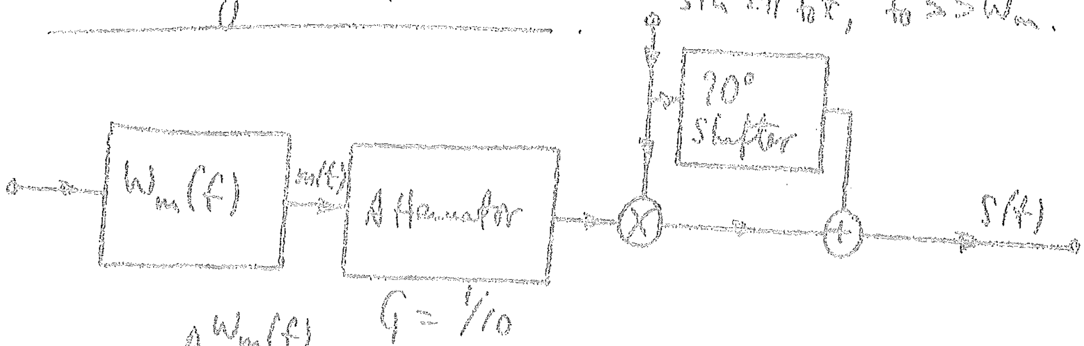
Thus, the *average* $\frac{|\bar{S}_i-\bar{S}_k|}{\sqrt{2N_0}}$ for set (a) is approximately

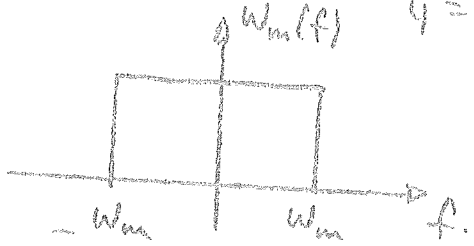
$$\sqrt{\frac{E_s}{(1.8)N_0}}.$$

If it were $\sqrt{\frac{E_s}{2N_0}}$ we would have *exactly* a 3-dB degradation in performance (the signal energy would be effectively halved). Thus, because "1.8 is close to 2" we have approximately 3-dB poorer performance with the signals of set (a) than (b).

Solutions of 8.16, 8.19.

(8.16).



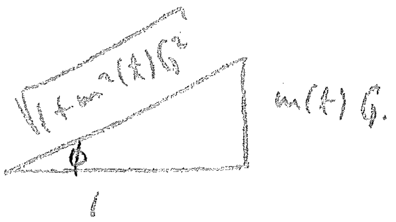


(a). $S_m(t) = m(t) \cdot G \sin 2\pi f_0 t + \sin(2\pi f_0 t + 90°)$.

$$= m(t)\, G \sin 2\pi f_0 t + \cos 2\pi f_0 t.$$

$$= \sqrt{1 + m^2(t) G^2}\, \cos(2\pi f_0 t - \varphi).$$

$\varphi = \tan^{-1}(m(t) G)$.



For $|m(t) G| << 1$,

$\tan^{-1} G\, m(t) \simeq G\, m(t)$

and $\sqrt{1 + m^2(t) G^2} \simeq 1$.

$\therefore$ $S_m(t) = \cos(2\pi f_0 t - G\, m(t))$ — Phase modulation.

(b) For F.M. : $S_m(t)_{FM} = A\sqrt{2}\, \cos 2\pi\left(f_0 t + W_1 \int m(t)\, dt\right)$.

For PM: $S_m(t)_{PM} = A\sqrt{2}\cos 2\pi\left(f_0 t + W_2\int m'(t)\,dt\right)$

$= A\sqrt{2}\cos 2\pi\left(f_0 t + W_2\, m(t)\right).$

For sinusoidal signals, $m(t) = \left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}(2\pi W_m t)$, and in order β_{FM} correspond in meaning to β_{PM} we must have

$$S_m(t)_{FM} = A\sqrt{2}\cos 2\pi\left(f_0 t + \beta_{FM}\left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}(2\pi W_m t)\right).$$

$$S_m(t)_{PM} = A\sqrt{2}\cos 2\pi\left(f_0 t + \beta_{PM}\left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}(2\pi W_m t)\right).$$

But $\beta_{FM} = \frac{W_1}{W_m}$ and $\beta_{PM} = 2\pi W_2$.

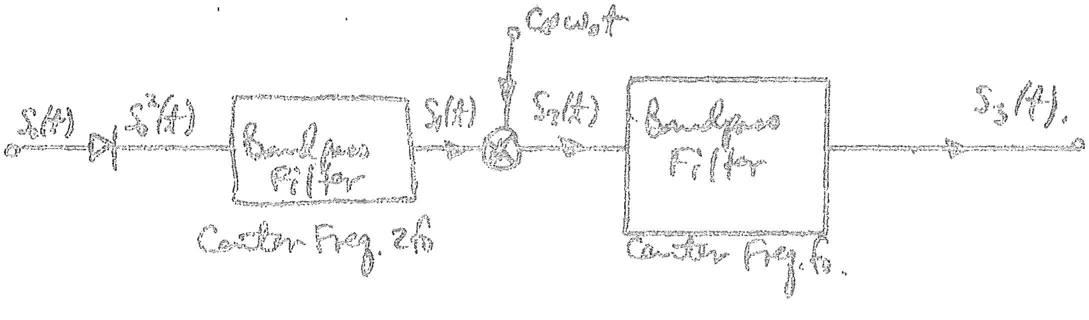
For $S(t)$, $\boxed{\beta = \frac{G}{2\pi} \times 2\pi = G. \approx \frac{1}{10}}$

(b) A better reason for calling $2\pi W_2$ the <u>modulation index</u> stems from 8.151b wherein the mean-square output noise of the PM system is

$$\overline{n^2(t)} = \frac{1}{(2\pi W_2)^2} \cdot \frac{N_0}{2E_s}.$$ Thus,

$2\pi W_2$ plays the same role in PM as does β in FM (8.186). Thus, $2\pi W_2$ is the <u>modulation index</u> of a PM system.

(c)



$s_0(t) = \cos 2\pi(f_0 t + \varphi)$, where $\varphi = G\, m(t)$.

$= \cos(\omega_0 t + \beta_0 m(t))$.

$s_0^2(t) = \frac{1}{2}\left[1 + \cos(2\omega_0 t + 2\beta_0 m(t))\right]$.

$s_1(t) = \frac{1}{2}\cos(2\omega_0 t + 2\beta_0 m(t))$.

$s_2(t) = \frac{1}{4}\left[\cos(3\omega_0 t + 2\beta_0 m(t)) + \cos(\omega_0 t + 2\beta_0 m(t))\right]$.

$s_3(t) = \frac{1}{4}\cos(\omega_0 t + 2\beta_0 m(t))$.

$\therefore\ \beta_1 = 2\beta_0 = 4\pi G$.

In general, $\beta_K = 2^K \beta_0$, where K = # of stages.

(d). <u>Required</u>: $\overline{n^2(t)} = 10^{-4}$.

Given: $G = 1/10$, $W_m = 3{,}000$ Hz., $\frac{N_0}{2} = 10^{-11}$ J., $P_s = 10^{-6}$ W.

For max. likelihood reception of PM signals,

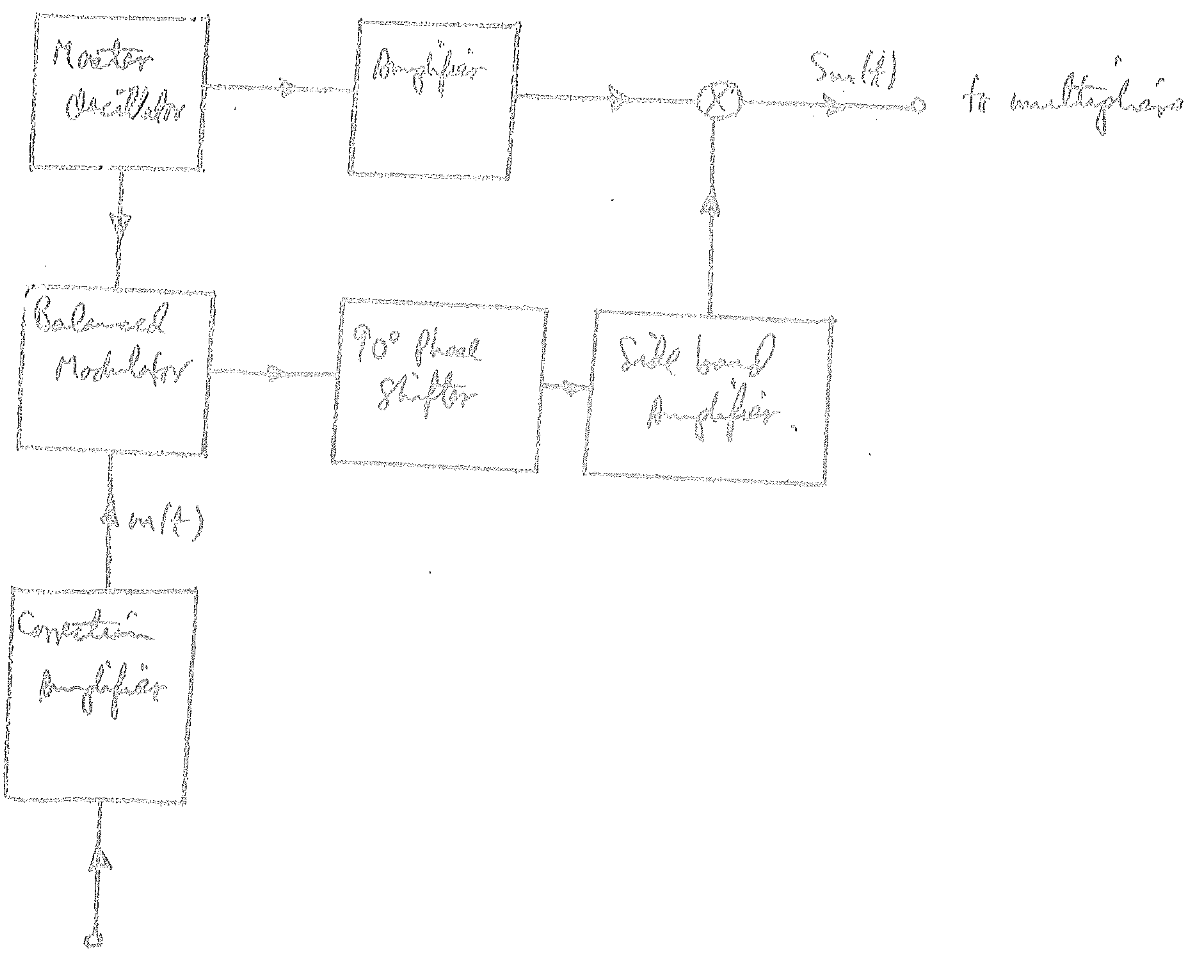
$$\overline{n^2(t)} = \frac{1}{\beta_K^2}\cdot\frac{N_0}{2E_s} \qquad (8.156b \text{ or } 8.157c).$$

From 8.158, $E_S = \frac{P_s}{2W_m} = \frac{10^{-6}}{6000} = 1.67\cdot 10^{-10}$ J.

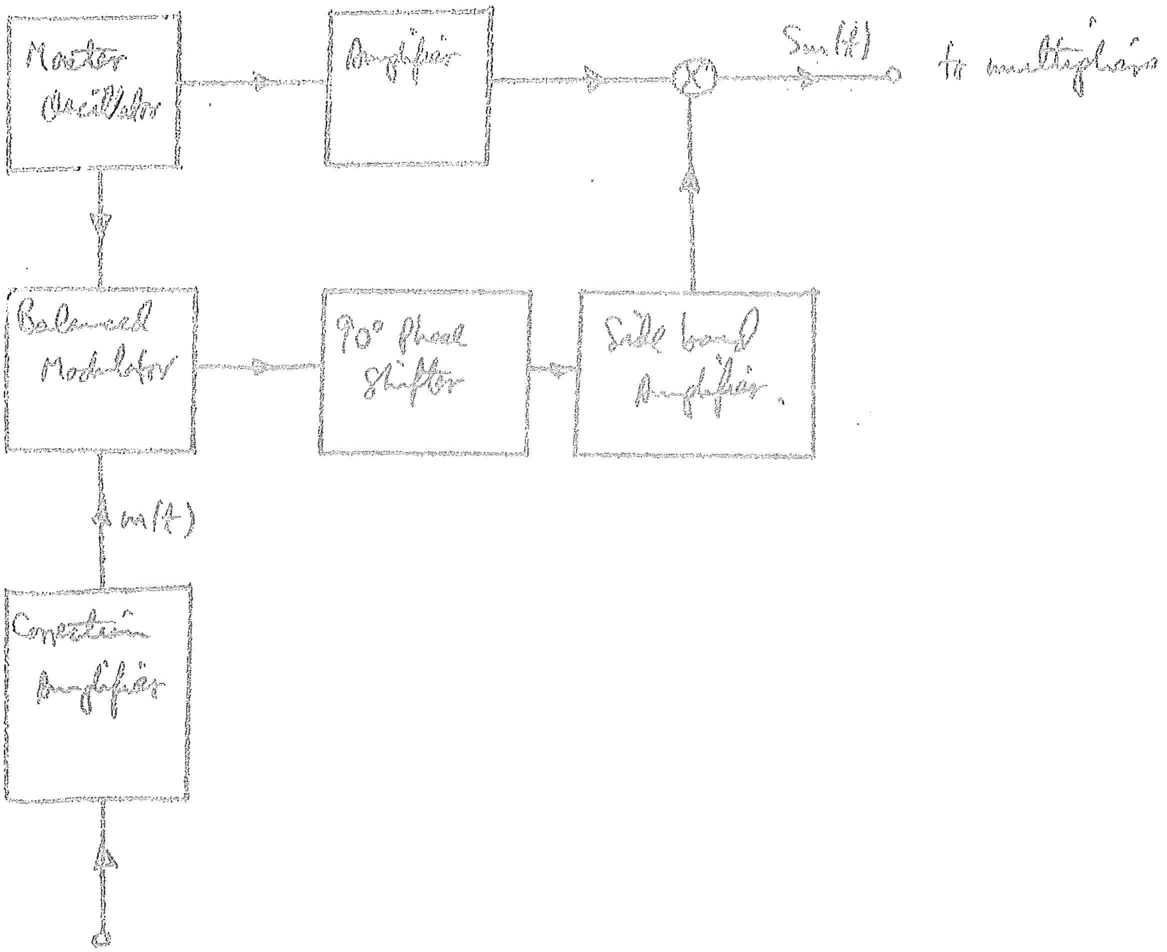
$\therefore\ 10^{-4} \geq \frac{1}{\beta_K^2}\cdot\frac{10^{-11}}{1.67\cdot 10^{-10}} = 0.06\cdot\frac{1}{\beta_K^2} \Rightarrow \beta_K^2 \geq \frac{6\cdot 10^{-2}}{10^{-4}} = 600.$

$\therefore\ \beta_K \approx 25$, or $2^K(0.628) \approx 25 \Rightarrow 2^K \approx 40$ or $\boxed{K = 6 \text{ stages}}$

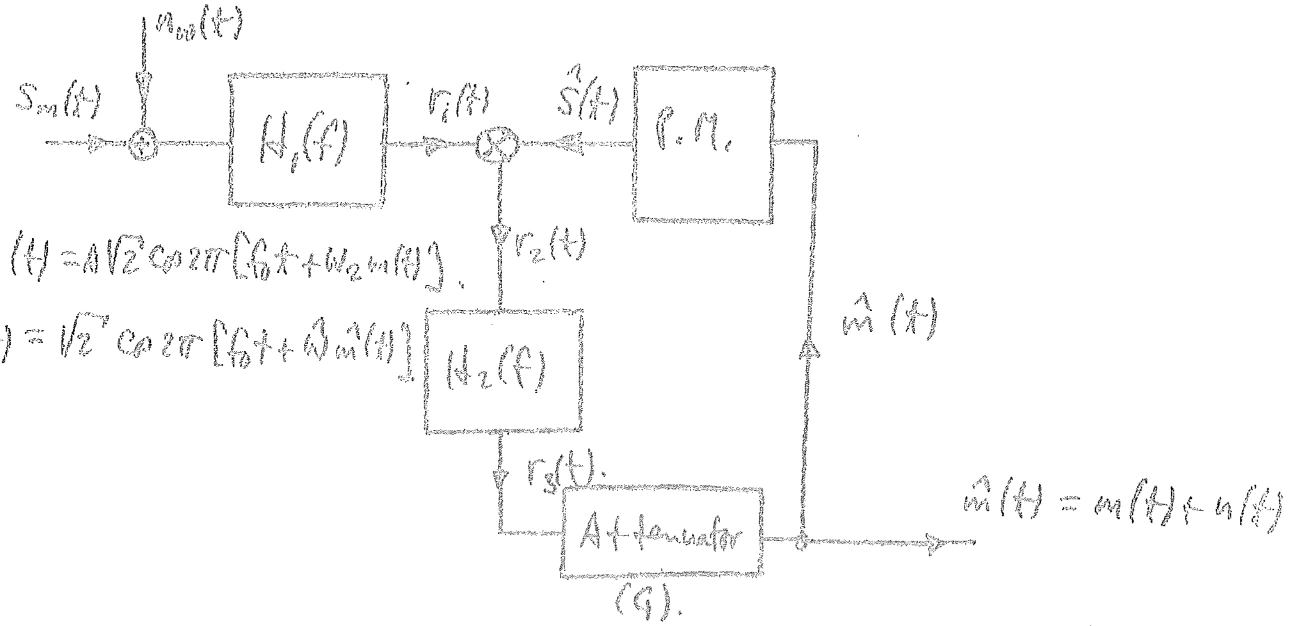
(e). Armstrong's FM-system.



(c). Armstrong's FM - System.



(8.19)



$s_m(t) = A\sqrt{2}\cos 2\pi[f_0 t + W_2 m(t)]$.

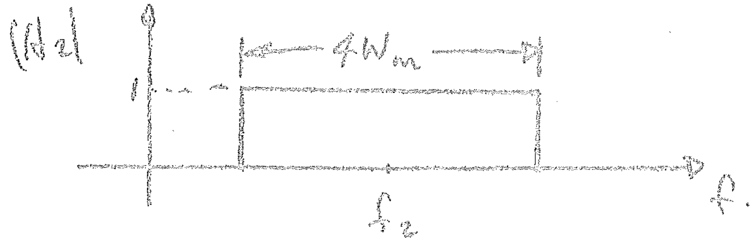
$\hat{s}(t) = \sqrt{2}\cos 2\pi[f_0 t + \hat{W}\hat{m}(t)]$

(a) $\quad s_m(t) = B\cos(\omega_0 t + \theta)$

$\hat{s}(t) = \hat{B}\cos(\omega_0 t + \hat{\theta})$.

$r_2(t) = s(t)\hat{s}(t)$. $\qquad$ (note $n_w(t) = 0 \Rightarrow r_1(t) = s_m(t)$).

$= \frac{B\hat{B}}{2}[\cos(2\omega_0 t + \theta + \hat{\theta}) + \cos(\theta - \hat{\theta})]$.



H_2 must be narrow enough to block the carrier ($2\omega_0$) but wide enough to pass $\cos(\theta - \hat{\theta})(t)$. Thus $H_2(f)$ must be as shown above.

$r_3(t) = \frac{B\hat{B}}{2}\cos[\theta(t)-\hat{\theta}(t)]$, and $\hat{m}(t) = \frac{B\hat{B}G}{2}\cos[\theta(t)-\hat{\theta}(t)]$.

$$= \frac{GB\hat{B}}{2}\cos[w_2 m(t) - \hat{w}\hat{m}(t)].$$

$$= \frac{GB\hat{B}}{2}\sin(\pi/2 + w_2 m(t) - \hat{w}\hat{m}(t)].$$

$$\simeq \frac{GB\hat{B}}{2}\left[\frac{\pi}{2} + w_2 m(t) - \hat{w}\hat{m}(t)\right]$$, if the argument of the sine function is small, i.e.

$$\hat{w}\hat{m}(t) \simeq \frac{\pi}{2} + w_2 m(t).$$

Hence, for this "linear approximation", we have

$$\hat{m}(t)\left[1 + \frac{G\cancel{B}\hat{B}\hat{w}}{2}\right] = \frac{G\cancel{B}\hat{B}}{2}\left(\frac{\pi}{2} + w_2 m(t)\right].$$

and $$\boxed{\hat{m}(t) = \frac{\frac{GB\hat{B}w_2}{2}}{1+\frac{GB\hat{B}\hat{w}}{2}}m(t) + \frac{\frac{GB\hat{B}\pi}{4}}{1+\frac{GB\hat{B}\hat{w}}{2}}}$$

Upon setting $GB\hat{B} = \frac{2}{w_2 - \hat{w}}$ or $\boxed{G = \frac{2}{B\hat{B}(w_2-\hat{w})}}$

we get $\hat{m}(t) = m(t) + K$, where K is a constant. Hence, up to an additive constant (which can be eliminated by a differentiator) $\hat{m}(t) = m(t)$.

1. For simplicity let $m(t) = 0$. Then

$$r_1(t) = s_m(t) + n_1(t)$$

$$n_1(t) = n_c(t)\sqrt{2}\cos\omega_0 t + n_s(t)\sqrt{2}\sin\omega_0 t.$$

Filtered Gaussian white noise, $n_c(t)$ is the <u>in-phase</u> and $n_s(t)$ is the <u>out-of-phase</u> component of noise,

$$r_1(t) = [A + n_c(t)]\sqrt{2}\cos\omega_0 t + n_s(t)\sqrt{2}\sin\omega_0 t.$$

$$= a(t)\sqrt{2}\cos[\omega_0 t + \varphi(t)].$$

$$\varphi(t) = \tan^{-1}\frac{-n_s(t)}{A + n_c(t)} \simeq \frac{-n_s(t)}{A}$$ for weak noise,

$$a(t) = \left\{[A + n_c(t)]^2 + n_s^2(t)\right\}^{1/2} \simeq A.$$

$$r_2(t) = \hat{s}(t)\cdot r_1(t). = \frac{\sqrt{2}\,a(t)\sqrt{2}}{2}\left[\cos(2\omega_0 t + \varphi(t) + \hat{\theta}(t)) + \cos(\varphi(t) - \hat{\theta}(t))\right]$$

$$= a(t)\left[\cos(2\omega_0 t + \varphi(t) + \hat{\theta}(t)) + \cos(\varphi(t) - \hat{\theta}(t))\right].$$

$$= A\left[\cos(2\omega_0 t + \varphi(t) + \hat{\theta}(t)) + \cos(\varphi(t) - \hat{\theta}(t))\right].$$

$$r_3(t) = A\cos(\varphi(t) - \hat{\theta}(t)),$$

$$\hat{m}(t) = G A\cos(\varphi(t) - \hat{\theta}(t)),$$

$$\hat{m}(t) = m(t) + n(t) = n(t),$$ (because $m(t) = 0$, by assumption).

Hence, $u(t) = GA\cos(\varphi(t) - \theta(t))$.

$$\simeq GA\left(\frac{\pi}{2} + \varphi(t) - \theta(t)\right).$$

$$= GA\left(\frac{\pi}{2} + \left(-\frac{n_s(t)}{A}\right) - 2\pi\hat{w}\,\hat{m}\right)$$

$$= GA\left(\frac{\pi}{2} - \frac{n_s(t)}{A} - 2\pi\hat{w}\,u\right).$$

$$\therefore \quad u(t) = \frac{GA\cdot\frac{\pi}{2} - G\,n_s(t)}{1 + 2\pi\,\hat{w}\,GA}.$$

(4-7) Given $\int_0^T x^2(t)\,dt = \int_0^T y^2(t)\,dt = 16\ j,$

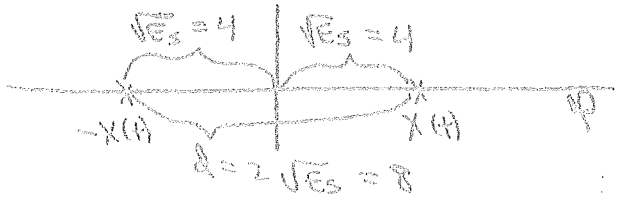
and $\int_0^T x(t)\,y(t)\,dt = 0$

This information says (1) The total energy in $x(t)$ = the total energy in $y(t) = E_s = 16\ j.$

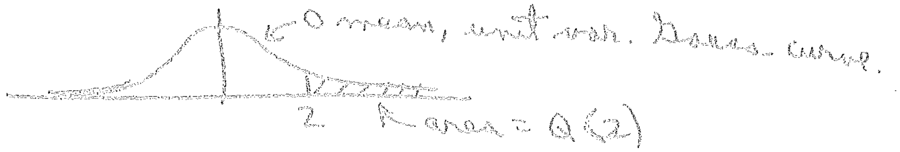
(2) $x(t)$ and $y(t)$ are orthogonal

Assuming equally likely messages, and $S_n(f) = \frac{\eta_0}{2} = 4\ \frac{\text{watt-sec}}{\text{cycle}}$:

(a) Let $x(t)$ and $-x(t)$ be represented in signal space as:



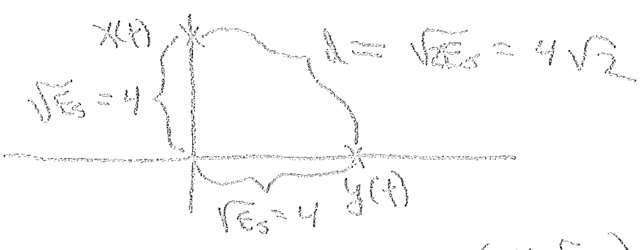
$$P[\mathcal{E}] = \tfrac{1}{2} P[\mathcal{E} \mid x(t)] + \tfrac{1}{2} P[\mathcal{E} \mid -x(t)] = Q\left(\frac{d}{\sqrt{2\eta_0}}\right) = Q\left(\frac{8}{\sqrt{2(8)}}\right)$$



$= Q(2)$

From table books: $\underline{P[\mathcal{E}] = Q(2) = .0227}$

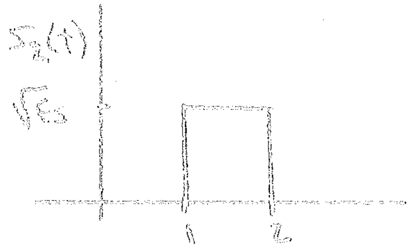
(b) $x(t)$ and $y(t)$ are represented in signal space as:

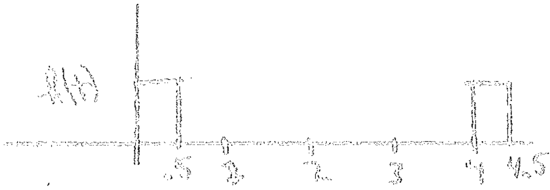


As before, $P[\mathcal{E}] = Q\left(\frac{d}{\sqrt{2\eta_0}}\right) = Q\left(\frac{4\sqrt{2}}{\sqrt{2(8)}}\right) = Q(\sqrt{2})$

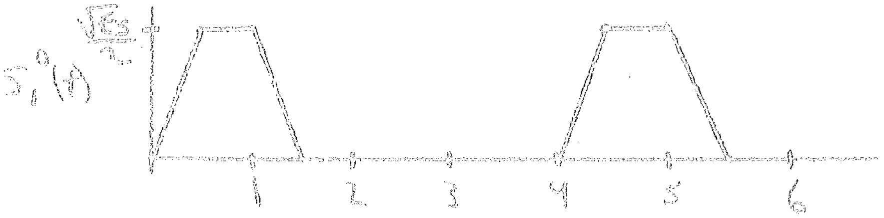
From table books: $\underline{P[\mathcal{E}] = Q(\sqrt{2}) \cong .0788}$

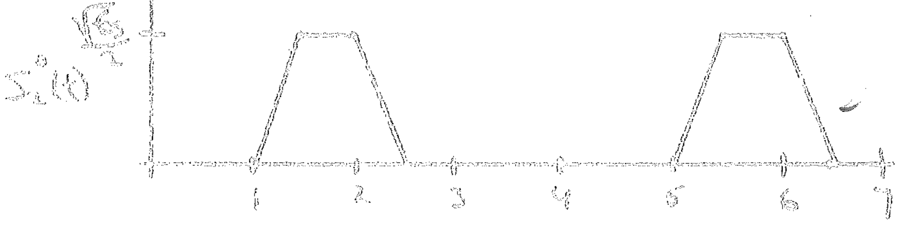
EE 592 Homework Prob 1.1





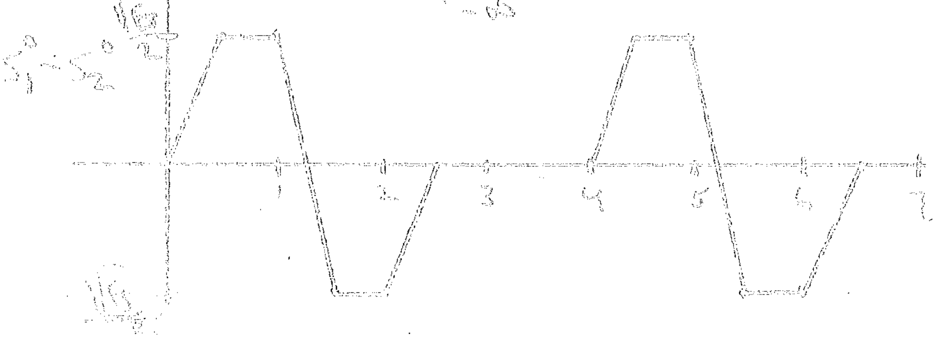
$s_1^0(t) = s_1(t) * h(t)$ and $s_2^0(t) = s_2(t) * h(t)$ (shown below)

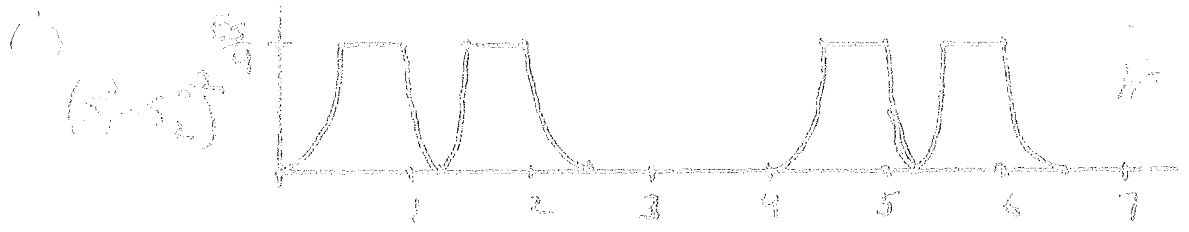




Now $P[E] = Q\left(\frac{d}{\sqrt{2N_0}}\right)$ where d is the distance between s_1^0 and s_2^0 in an orthogonal space. This distance can be written in terms of the energy of the difference between $s_1^0(t)$ and $s_2^0(t)$ as follows:

$$d = \sqrt{\int_{-\infty}^{\infty}(s_1^0(t) - s_2^0(t))^2\,dt}$$





$$d^2 = \int (s_1^2(t) - s_2^2(t))^2\,dt = 4\left[\frac{1}{3}\times\frac{1}{2}\times\frac{E_s}{4} + \frac{1}{2}\times\frac{E_s}{4} + \frac{1}{3}\times\frac{1}{4}\times\frac{E_s}{4}\right]$$

$$\therefore\ d^2 = \frac{3E_s}{4} \text{ or } d = \sqrt{\frac{3E_s}{4}}$$

$$\underline{\underline{P[\epsilon]}} = Q\left(\frac{d}{\sqrt{2n_0}}\right) = Q\left(\frac{\sqrt{\frac{3E_s}{4}}}{\sqrt{2n_0}}\right) = Q\left(\sqrt{\frac{3E_s}{8n_0}}\right) = \underline{\underline{Q\left(\sqrt{\frac{15}{8}}\right)}}$$

because $\frac{E_s}{N_0}$ is given as 5.

$Q(\sqrt{1.89)5})$

$5\frac{\sqrt{E_s}}{\sqrt{48}} + \frac{\sqrt{105 E_s}}{\sqrt{24}}$

$\frac{25E_s + 210E_s}{48}$

$\sqrt{\frac{3E_s}{4}} - \sqrt{\frac{E_s}{48}} \quad \sqrt{\frac{36E_s}{48}} - \sqrt{\frac{E_s}{48}}$

$\frac{\sqrt{36E_s}}{\sqrt{48}} - \sqrt{E_s} = 6\frac{\sqrt{E_s} - \sqrt{E_s}}{\sqrt{48}} \quad \frac{\sqrt{5}E_s}{\sqrt{48}}$

$.75$
$.50$
1.25

105
12
210
105
1260

1.25

$\sqrt{\frac{5}{48}E_s + \frac{105}{4}E_s}$

$\frac{5}{48} + 105$

$\sqrt{\frac{1265}{48}}$

$(3.0-2t)(2t-2)$

$48\overline{)1.000}$

$\frac{3}{5} + \frac{1}{16}\ \frac{3/5\ E_s}{\sqrt{E_s} - \frac{1}{4}\sqrt{\frac{3}{5}}}$
$2N_0$

$\sqrt{\frac{235}{48}}E_s$

$\frac{3}{E_s} + \frac{\frac{1}{16}3/5}{E_s^{3/2}}\ \frac{1}{4}\sqrt{E_s} - \sqrt{E_s}$

$48\overline{)1.00}$ .010
96
40

$\frac{2E_s}{2}\int(3.0-2t)$

.9

$48\overline{)235}$

$\sqrt{105}$

$\sqrt{36E_s}$

$24\overline{)1.00}$ .04
96
40

$48\overline{)1.00}$ .04

$48\overline{)1265}$ 2
96
305

1.37
1.37
924
4110
13700
18734

$96\overline{)1000}$ .010
96
40

$24\overline{)1.00}$ .041
96
40